# ANALYSIS OF A MATHEMATICAL MODEL ON THE SPREAD AND CONTROL OF HIV/AIDS WITH DIFFERENT MODES OF TRANSMISSIONS AND INFLOW OF IMMIGRANTS IN ETHIOPIA

*Thesis Submitted to Debre Berhan University for the Award of the Degree of*

DOCTOR OF PHILOSOPHY

IN

MATHEMATICAL MODELING

By

TIBEBU TULU GUYA, M.Sc

Under the Advisor of

Dr. TEMESGEN TIBEBU MEKONNEN
Associate Professor of Mathematical Modeling
Department of Mathematics

DEBRE BERHAN UNIVERSITY, DEBRE BERHAN
ETHIOPIA
DECEMBER 17, 2020

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## EXAMINERS' APPROVAL SHEET

We, the undersigned, members of the Board of Examiners of the final open defense have read and evaluated the thesis prepared by **Tibebu Tulu Guya** entitled **" Analysis of a Mathematical Model on the Spread and Control of HIV/AIDS with Different Modes of Transmissions and Inflow of Immigrants in Ethiopia "** and examined the candidate. This is therefore to certify that the thesis has been accepted in partial fulfillment of the requirements for the degree of doctor of philosophy in mathematical modeling complies with regulation of the University and meets the accepted standards with respect to originality.

| **Name** | **Signature** | **Date** |
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| **Dr. Temesgen Tibebu** (Main Adviser) | _____ | 30-11-2020 |
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Final approval and acceptance of the thesis is contingent upon the submission of the final copy of the thesis to the College of Graduate Studies (CGS) through the Department Graduate Committee (DGS) of the candidate's.

Date _____

## FINAL THESIS APPROVAL FORM

As members of the Board of Examiners of the final PhD open defense, we certify that we have read and evaluated the thesis prepared by **Tibebu Tulu Guya** entitled **" Analysis of a Mathematical Model on the Spread and Control of HIV/AIDS with Different Modes of Transmissions and Inflow of Immigrants in Ethiopia "** and recommend that it be accepted as fulfilling the thesis requirement for the degree of doctor of philosophy in mathematical modeling.

| **Name** | **Signature** | **Date** |
|---|---|---|
| **Prof. Dr. Abdul Vahid Hasmani** (External Examiner) | A.H.Hasmani | 30-11-2020 |
| **Prof. Dr. Koya Purnachandra Rao** (Internal Examiner) | | 30-11-2020 |

Final approval and acceptance of the thesis is contingent upon the submission of the final copy of the thesis to the CGS through the DGC of the Mathematics Department.

| **Name** | **Signature** | **Date** |
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| _____ (DGS) | _____ | _____ |

Certification of the final Thesis

I hereby certify that all the corrections and recommendations suggested by the Board of Examiners are incorporated into the final Thesis entitled **" Analysis of a Mathematical Model on the Spread and Control of HIV/AIDS with Different Modes of Transmissions and Inflow of Immigrants in Ethiopia "** by Tibebu Tulu Guya.

| **Name** | **Signature** | **Date** |
|---|---|---|
| _____ (CGC) | _____ | _____ |

# Declaration

I hereby declare that this thesis entitled " Analysis of a Mathematical Model on the Spread and Control of HIV/AIDS with Different Modes of Transmissions and Inflow of Immigrants in Ethiopia " is my own work and that, to the best of my knowledge and belief it contains no material previously published or written by another person or material which has been accepted for the award of any other degree or diploma of the university or other institute of higher learning.

Tibebu Tulu Guya

Debre Berhan, Ethiopia

DECEMBER 17, 2020

# Certificate

This is to certify that the thesis entitled " Analysis of a Mathematical Model on the Spread and Control of HIV/AIDS with Different Modes of Transmissions and Inflow of Immigrants in Ethiopia " submitted by Tibebu Tulu Guya towards the award of the degree of doctor of philosophy in Mathematical Modeling in the department of Mathematics, Debre Berhan University, Debre Berhan, Ethiopia is a genuine record of the work carried out by him under my supervision and guidance.

Dr. Temesgen Tibebu Mekonnen
Associate Professor of Mathematical Modeling
Department of Mathematics, Debre Berhan University
Debre Berhan, Ethiopia
DECEMBER 17, 2020

# Acknowledgments

After an intensive period of learning, today is the day: writing this note of thanks is the finishing touch on my thesis. It has been a period of intense learning for me, not only in the scientific arena, but also on a personal level. Writing this thesis has had a big impact on me. I would like to reflect on who have supported and helped me so much throughout this period. First and foremost I would like to thank God. He has given me the power to believe in myself and pursue my dreams. I could never have done this without the faith I have in you, the Almighty. Second, I must express gratitude towards my advisor, Dr. Temesegen Tibebu. I appreciate his guidance, dedication and exemplary academic standards for the preparation of this thesis. Next, I have to thank my parents for their love, their wise counsel and sympathetic ear and support throughout my life. They are always there for me. My mother and father, who taught me many valuable lessons in life and always gave their best advice. Their encouragement and prayers have made me strong enough to face all the problems in this thesis.

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# Dedication

To my parents and my family.

# List of Publications

**Tibebu Tulu Guya and Temesgen Tibebu Mekonnen**. *"Treatment and Inflow Infective Immigrants on the Dynamics of HIV/AIDS"* IOSR Journal of Mathematics (IOSR-JM)e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 4 Ser. II (Jul – Aug 2019), PP 71-82.

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# Acronyms

| | |
|---|---|
| AIDS | Acquired Immune Deficiency Syndrome |
| ART | Anti Retroviral Therapy |
| CDC | Center for Disease Control |
| CSW | Commercial Sex Workers |
| DFE | Disease Free Equilibrium |
| DHS | Demographic Health Survey |
| FHAPCO | Federal HIV/AIDS Prevention and Control Office |
| GBV | Gender Based Violence |
| HBV | Hepatitis B Virus |
| HCT | HIV Counseling and Testing |
| HIV | Human Immunodeficiency Virus |
| MATLAB | Matrix Laboratory |
| NGO | Non Government Organization |
| ODE | Ordinary Differential Equation |
| PDE | Partial Differential Equation |
| PEPFAR | President's Emergency Plan for AIDS Relief |
| SIV | Simian Immunodeficiency Virus |
| SNNP | Southern Nations, Nationalities and Peoples' |
| STD | Sexual Transmitted Disease |
| WHO | World Health Organization |

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# Chapter 1

# Introduction

HIV/AIDS is a global infectious disease. As of 2016, approximately 36.7 million people have HIV worldwide with the number of new infections that year being about 1.8 million [52]. This is down from 3.1 million new infections in 2001 [19]. Slightly over half the infected population are women and 2.1 million are children. It resulted in about 1 million deaths in 2016, down from a peak of 1.9 million in 2005 [52]. In 2008 in the United States approximately 1.2 million people were living with HIV, resulting in about 17,500 deaths. The US Centers for Disease Control and Prevention estimated that in 2008 20% of infected Americans were unaware of their infection. As of 2016 about 675,000 people have died of HIV/AIDS in the USA since the beginning of the HIV epidemic. In the United Kingdom as of 2015 there were approximately 101,200 cases which resulted in 594 deaths. South & South East Asia is the second most affected; in 2010 this region contained an estimated 4 million cases or 12% of all people living with HIV resulting in approximately 250,000 deaths. Approximately 2.4 million of these cases are in India [52]. Sub-Saharan Africa is the region most affected. In 2010, an estimated 68% (22.9 million) of all HIV cases and 66% of all deaths (1.2 million) occurred in this region. This means that about 5% of the adult population is infected and it is believed to be the cause of 10% of all deaths in children. Here in contrast to other regions women compose nearly 60% of cases. South Africa has the largest population of people with HIV of any country in the world at 5.9 million. Life expectancy has fallen in the worst-affected countries due to HIV/AIDS; for example, in 2006 it was estimated that it had dropped from 65 to 35 years in Botswana. Mother-to-child transmission, as of 2013, in Botswana and South Africa has decreased to less than 5% with improvement in many other African nations due to improved access to antiretroviral therapy [52]. Accurate data on the number of Ethiopians who have

died of HIV/AIDS in the last three decades are hard to know, but according to an estimate by the Federal HIV/AIDS Prevention and Control Office (FHAPCO), there were on average 19,743 deaths every year, which left behind about 247,250 children orphaned [42]. Following its official discovery in 1984, HIV/AIDS has seen a rapid spread all over Ethiopia, putting the lives of millions at risk directly or indirectly. Owing to lack of awareness, similar trend is seen in many countries in Africa and Asia. Like many of these countries, Ethiopia's fight against HIV/AIDS is one of the few highly paid for projects by western donors, which includes the over US$2 billion contribution by the U.S. government through its program called 'President's Emergency Plan for AIDS Relief (PEPFAR), making it the ever largest donation coming to the country for HIV/AIDS purpose. Owing to similar coordinated efforts, the spread of the virus has seen a decline over the last two decades especially in urban areas. As data from FHAPCO indicates that there are over 718,550 people living with HIV in Ethiopia alone, a little over 1.18% of the population [42]. The 2016 Ethiopian Demographic Health Survey (DHS) reveals that around 56% of the women and 55% of the men among the surveyed household have never been tested for HIV, an indication the current number of HIV positives in the country could be a lot more had all the population been tested. And, despite the existence of the large number of people living with HIV/AIDS, only 72% of them are thought to be aware that they are living with the virus; the remaining 28% think they are not infected. When measuring the prevalence of HIV women tend to be more vulnerable than men. Of all the HIV positives in Ethiopia, 39% are men while women account for the remaining 61%, of which 25% of are commercial sex workers [42]. According to the FHAPCO, 27,288 people were known to have been infected by HIV during the 2009 Ethiopian calendar; 16,021 (59%) were women whereas 11,267 (41%) were men. Among the three million pregnant women who are receiving medical follow up currently, around 27,000 of them are HIV positives [42]. Gender-based violence (GBV) is another major factor contributing to increase numbers among women at risk of contracting HIV. The 2016 Ethiopian DHS report shows that among all the gender based violence in 2016, 7% of them were sexual, and one in 10 women among the surveyed experienced sexual violence. The report also mentions divorced, separated and widowed women as the most affected by sexual violence, compared to married women [42]. HIV/AIDS has become a chronic rather than an acutely fatal disease in many areas of the world [62].

## 1.1 Research Problem

One STD that many people are worried about getting is HIV/AIDS which is nowadays considered as the greatest public health disaster of modern time. Its progression has challenged the global population for decades. Through mathematical modeling, researchers have studied different interventions on the HIV pandemic, such as treatment, education, condom use, and those focuses on different compartmental models with emphasis on the effect of public health education. As a motivation for this research, it is important not to let our arms down in our efforts to prevent and control the HIV/AIDS epidemics in our countries. If mathematical models based on the different mode of transmission mechanism of HIV/AIDS might help the medical and scientific community to understand better how the disease spreads in the community then we have to support it. Even though the actual data needed for the models might not be accurate or even available, such modeling is still vital in investigating how changes in the various assumptions and parameter values affect the course of the epidemic. So we would like to analyze a deterministic mathematical model analysis on the spread and control of HIV/AIDS under different modes of transmissions and inflow of immigrants. In this thesis we responded the following research questions.

- What is the most influential parameter that helps the spread of HIV/AIDS in the community?
- What is the most influential parameter that helps us to control the spread of HIV infection based on the real data taken from Ethiopia?

## 1.2 Objectives of the study

### 1.2.1 Main Objective of the Study

In this thesis the main objective is analyzing a Mathematical model on different modes of transmission of HIV infection, age structure and inflow immigrants in Ethiopia.

### 1.2.2 Specific Objectives of the Study

The specific aims of this study are as follows:

- To identify the model assumptions on the dynamics of HIV infection.
- To construct flow charts based on the assumptions.
- To develop a dynamical system of HIV/AIDS infection.
- To determine equilibrium points for the dynamical system.
- To analyze stability of equilibrium points.
- To identify the basic reproduction number.
- To determine the control parameters on the spread and control of HIV/AIDS infection.
- To identify the most sensitive parameters.

## 1.3 Significance of the study

The goal of HIV infection control programme is to decrease morbidity and mortality due to this disease and prepare planning to control transmission of HIV infection in the community. Understanding the dynamic of HIV/AIDS is a key to the control of the epidemic. The study will give insight into dynamics of HIV which is crucial in the control of the disease. Further, the findings of this study will be of great benefit to the public health sector, the community and NGOs. It will be also helpful in policy formulation, planning, budgeting, resource allocation and making appropriate decisions in control and prevention of the diseases and able to prescribe proper interventions. The study will also add to the existing body of knowledge on mathematical application in the field of epidemiology.

This thesis is organized as follows:

First, in Chapter two, we presented the literature review and related materials.

Next, in Chapter three, we explain and discuss about the methodology we used in the study.

In Chapter four, the focus is developing a mathematical model for treatment and inflow infective immigrants on the Dynamics of HIV/AIDS and analyzed analytically. Here finding the reproduction number and stability analysis of equilibrium points are discussed.

In Chapters five, we investigate a mathematical model analysis on Dynamics of HIV/AIDS with age structure and inflow immigrants in Ethiopia. The focus of these chapter is finding the reproduction number and stability analysis of equilibrium points.

In Chapter six, we investigate numerical simulation and sensitivity analysis of a mathematical

model developed and discussed under Chapter four. Here we also identify the most sensitive parameters.

In Chapter seven, we investigate numerical simulation, parameter estimation and sensitivity analysis of a mathematical model developed and discussed under chapter five. Here we also identify the most sensitive parameters using data taken from Ethiopia.

Finally, we conclude and recommend the thesis in Chapter 8.

# Chapter 2

# Literature Review

## 2.1 Origin of HIV

Both HIV-1 and HIV-2 are believed to have originated in non-human primates in West-central Africa and were transferred to humans in the early 20th century [102]. HIV-1 appears to have originated in southern Cameroon through the evolution of SIV (cpz),a simian immunodeficiency virus (SIV) that infects wild chimpanzees (HIV-1 descends from the SIVcpz endemic in the chimpanzee subspecies Pan troglodytes troglodytes) [37]. The closest relative of HIV-2 is SIV(smm), a virus of the sooty mangabey, an Old World monkey living in coastal West Africa (from southern Senegal to western Côte d'Ivoire) [96]. There is evidence that humans who participate in bush meat activities, either as hunters or as bush meat vendors, commonly acquire SIV [61]. However, SIV is a weak virus which is typically suppressed by the human immune system within weeks of infection. It is thought that several transmissions of the virus from individual to individual in quick succession are necessary to allow it enough time to mutate into HIV. Furthermore, due to its relatively low person-to-person transmission rate, SIV can only spread throughout the population in the presence of one or more high-risk transmission channels, which are thought to have been absent in Africa before the 20th century. An alternative view holds that unsafe medical practices in Africa after World War II, such as unsterile reuse of single use syringes during mass vaccination, antibiotic and anti-malaria treatment campaigns, were the initial vector that allowed the virus to adapt to humans and spread [18].

## 2.2 Stages of HIV infection

There are four main stages of HIV infection.

The first stage of HIV infection is called primary infection. Primary infection begins shortly after an individual first becomes infected with HIV. This stage lasts for a few weeks. During this period, individuals experience symptoms similar to the flu. Very few individuals seek treatment during this time, and those who do are usually misdiagnosed with a viral infection. Often, if an HIV test is performed, it will come back negative, since antibodies are not yet being produced by the individual's immune system [53]. Since antibodies have not yet developed, HIV continues to replicate and results in very high levels of the virus [4]. In the first few weeks after being infected, infected individuals are highly infectious. At this stage there is a large amount of HIV in the peripheral blood (the blood in the circulating system not in the lymphatic system, bone marrow, liver or spleen), around 106 copies of virus per $\mu l$ of blood. Antibodies and cytotoxic lymphocytes start being produced as a response to the virus which is known as seroconversion. At this stage about 20 percent of people who are HIV positive show symptoms which are not mild. However, the diagnosis of HIV infection is missed at this stage. Those who believe they have been exposed to HIV should repeat the test after six months.

In the second stage, individuals are free from any symptoms of HIV although there may be swollen glands. Levels of HIV in the blood are very low, but are detectable. If an HIV test is performed, it will show positive. While the individual is asymptomatic, the HIV in their blood is reproducing constantly. This stage lasts about ten years, but can be much longer or shorter depending on the individual and is characterized by a CD4+ count around 500 cells $\mu l$.

In the third stage, the immune system has become so damaged by HIV that symptoms begin to appear. As a result, it leads to greater CD4+ cell destruction and the immune system is not able to keep up with replacing the CD4+ cells that are lost. As the immune system fails, symptoms start to develop, Robertson [58]. Symptoms are typically mild at first, and then slowly become more severe. Opportunistic infections, infections that take advantage of the immune system's vulnerable state, begin to occur. These infections affect almost all the systems of the body and include both infections and cancers. Some common opportunistic infections include tuberculosis, cytomegalo virus, and shingles. In this stage HIV infection is often characterized by multi-system disease and infections in almost all body systems. Treatment for a specific infection or cancer is often carried out, however the main cause is the action of HIV as it attacks the immune system. Unless HIV itself can be reduced, immune suppression will continue to be

weaker.

In the fourth and final stage, a person is diagnosed as having AIDS. The progression to AIDS can be characterized by having a CD4+ count of 200 per ml or below, while the normal situation is around 1000 per ml. At this stage, the infected individual is likely to develop opportunistic infections in their respiratory system, gastro-intestinal system, central nervous system and on the skin as well. Once a person is diagnosed with AIDS, the AIDS status is permanent [58].

A blood test can determine if a person is infected with HIV, but if a person tests positive for HIV, it does not necessarily mean that the person has AIDS. A diagnosis of AIDS is made by a physician according to the CDC AIDS Case Definition. A person infected with HIV may receive an AIDS diagnosis after developing one of the CDC-defined AIDS indicator illnesses. A person with HIV can also receive an AIDS diagnosis on the basis of certain blood tests (CD4+ counts) and may not have experienced any serious illnesses.

Prognosis varies between people, and both the CD4 count and viral load are useful for predicted outcomes. Without treatment, average survival time after infection with HIV is estimated to be 9 to 11 years, depending on the HIV subtype [52]. After the diagnosis of AIDS, if treatment is not available, survival ranges lies between 6 and 19 months [73].

## 2.3 Epidemiology

One of the most important reasons that developed countries have become as productive as they are today is that the population remains healthy and disease free. This essential task is performed by each country's health department and is carried out by individuals known as epidemiologists. Without their efforts and their coordination with others in the medical field, it would be very difficult if not impossible to obtain current information regarding important diseases, methods of transmission, methods of control, and the like. Furthermore, information on the incidence or prevalence of diseases and statistics on morbidity and mortality rates, all of which are essential to physicians and other medical personnel to help control and understand diseases, would not be available without the efforts of the epidemiologists [13].

Ronald Ross (May 13, 1857 September 16, 1932) was an English physician. Ross was a pioneer in developing mathematical models for the study of epidemiology. Anderson Gray McKendrick (September 8, 1876 - May 30, 1943) Scottish physician and epidemiologist was another pioneer in the use of mathematical methods in epidemiology. McKendrick's career as a mathematical epidemiologist began in India. In 1914 he published a paper in which he gave equations for

the pure birth process and a particular birth-death process. After his return to Scotland he collaborated with W. O. Kermack on a notable series of papers. The first paper (1927) gave the differential equations for a deterministic general epidemic [99]. It is important to mention that modeling is very crucial in epidemiology since in most cases we cannot do experiments.

HIV is transmitted by three main routes: sexual contact, blood and blood products, and from mother to child. There is no risk of acquiring HIV if exposed to nasal secretions, saliva, sputum, sweat, tears, urine, or vomit unless these are contaminated with blood [63]. The most frequent mode of transmission of HIV is through sexual contact with an infected individual. Globally, the most common mode of HIV transmission is via sexual contacts between people of the opposite sex; however, the pattern of transmission varies among countries. As of 2014, most HIV transmission in the United States occurred among men who had sex with men [52]. The viral load of an infected person is an important risk factor in both sexual and mother-to-child transmission [8]. During the first 2.5 months of an HIV infection a person's infectiousness is twelve times higher due to the high viral load associated with acute HIV [27]. If the person is in the late stages of infection, rates of transmission are approximately eight fold greater [16]. The second most frequent mode of HIV transmission is via blood and blood products. Blood-borne transmission can be through needle-sharing during intravenous drug use, needle stick injury, transfusion of contaminated blood or blood product, or medical injections with unsterilized equipment.

Among blood & blood product transmissions, HIV is transmitted in about 93% of blood transfusions using infected blood [52]. In developed countries the risk of acquiring HIV from a blood transfusion is extremely low (less than one in half a million) where improved donor selection and HIV screening is performed; for example, in the UK the risk is reported at one in five million and in the United States it was one in 1.5 million in 2008. In low income countries, only half of transfusions may be appropriately screened (as of 2008), and it is estimated that up to 15% of HIV infections in these areas come from transfusion of infected blood and blood products, representing between 5% and 10% of global infections. Although rare because of screening, it is possible to acquire HIV from organ and tissue transplantation [52].

Unsafe medical injections play a significant role in HIV spread in sub-Saharan Africa. In 2007, between 12% and 17% of infections in this region were attributed to medical syringe use. The World Health Organization estimates the risk of transmission as a result of a medical injection in Africa at 1.2%. Significant risks are also associated with invasive procedures, assisted delivery, and dental care in this area of the world [97].

HIV can be transmitted from mother to child during pregnancy, during delivery, or through

breast milk, resulting in the baby also contracting HIV. This is the third most common way in which HIV is transmitted globally. In the absence of treatment, the risk of transmission before or during birth is around 20% and in those who also breastfeed 35% [4]. With appropriate treatment the risk of mother-to-child infection can be reduced to about 1% [4]. Preventive treatment involves the mother taking antiretroviral during pregnancy and delivery, an elective caesarean section, avoiding breastfeeding, and administering antiretroviral drugs to the newborn [58]. Antiretroviral when taken by either the mother or the infant decreases the risk of transmission in those who do breastfeed.
If a woman is untreated, two years of breastfeeding results in an HIV/AIDS risk in her baby of about 17%. Treatment decreases this risk to 1% to 2% per year. Due to the increased risk of death without breastfeeding in many areas in the developing world, the World Health Organization recommends either the mother and baby being treated with antiretroviral medication while breast feeding is continued or the provision of safe formula. Infection with HIV during pregnancy is also associated with miscarriage [52].

Epidemiology is the scientific study of epidemics and epidemic diseases, especially the factors that influence the incidence, distribution, and control of infectious diseases occurrence in human populations. It is possible to mathematically model the progress of most infectious diseases to discover the likely outcome of an epidemic or to help manage them by different control programs. In the early 20th century, mathematical methods were introduced into epidemiology by Ronald Ross [98], Anderson, Gray, McKendrick [113] and others. In the study of a disease, all of them had some quantitatively and qualitatively questions to answer: how many people have it?, where are these people?, how many new cases develop?, and how to control the disease?

During the development of epidemiology modeling in the population, deterministic (compartmental) models played a central role. Several papers are done with the deterministic mathematical model which has the central roles among those the deterministic model has been used in, [14], [31], [33], [36], [42], [43], [65] and so on. Such models divide the population into homogeneous sub-populations. The models that are labeled by $SI$, $SIS$, $SEIS$, and $SEIR$ are mostly used where the sub-populations are Susceptible, Exposed, Infected and Recovered or Removed.

In [31], Nyabadza et al. looked at a model of HIV/AIDS that examine the diminution in infection by promoting a change in sexual behavior through public health information campaigns and individuals with AIDS to abstain from sexual activities. They considered a sexually active population $N(t)$, at time $t$. In the absence of treatment and other post-exposure intervention, an adult individual's survival can be modelled using the four stage model of HIV disease progression, with the four stages corresponding to the WHO Clinical Staging System. Depending on the infection stage, they subdivide the population into subclasses (compartments): susceptible $S(t)$, asymptomatic infective $I_1(t)$ (infectious individuals who do not show symptoms of the disease), symptomatic infective $I_2(t)$ (HIV infected individuals who show symptoms of the disease) and those with full blown AIDS $A(t)$, who are assumed to be active in spreading the infection. The mode of transmission is assumed to be via heterosexual contacts as this represents the single major primary mode of HIV infection globally, especially in the worst affected regions of the world. Our sexually active population is thus given by $N(t) = S(t) + I_1(t) + I_2(t) + A(t)$. Each susceptible individual is considered to be equally likely to be infected by an infectious individual, i.e, the population mixes homogeneously. All parameters of the model are assumed positive. The recruitment rate of susceptible individuals is given by $\mu b$ where $b$ is total population and $\mu$ is natural death rate. The transfer rate from the asymptomatic compartment to the symptomatic compartment is $\sigma$. The removal rate of the symptomatic infective as they develop AIDS is given by $\rho$. The disease-related death rate is given by $\delta$. From assumptions they developed flow diagram and the dynamical system. Through analysis they obtained the reproduction number. The model can be used to quantify the role played by media campaigns and how they can possibly reduce the prevalence of the disease.

The HIV/AIDS epidemic in resource limited communities has been studied by Bhunu et al. in [17] they attempted to evaluate the impact that an increase in the fraction, through some social means, of sexually inactive HIV positive individuals has on the HIV/AIDS epidemic in sub-Saharan Africa. They developed a model of HIV/AIDS including separate classes of known HIV status and sexual activity levels, which are affected by HIV/AIDS education programs. The entire population is divided into the following sub-group compartments: the susceptible ($S$); the people who are HIV positive and do not know their status ($I_1$) ; the people who are HIV positive and know their status and reduce their risky sexual behavior as result knowing their status ($I_2$); the people who are HIV positive and know their status and have increased their risky sexual behaviour as a result of knowing their status ($I_3$);HIV positive people who

are sexually inactive ($I_4$); AIDS patients ($A$). It is assumed that the sexually inactive HIV positive individuals are no longer infectious. Their focus is mainly on testing and abstinence or voluntary withdrawal from sexual activity as soon as the individual is aware of his status. They also assume that all individual with the terminal form of AIDS are too sick to engage in sex and therefore do not contribute to the HIV/AIDS disease transmission dynamics. The total human population is given by $N(t) = S(t) + I_1(t) + I_2(t) + I_3(t) + I_4(t) + A(t)$. They note however that including the inactive classes $I_4$ and $A$ into the total population does not alter the dynamics of the model. Individuals are recruited into the human population through birth at a constant rate $\Lambda$. Susceptible ($S$) are infected with HIV following unprotected sexual contact with an infected individual at a rate $\lambda$ where $\lambda(t) = \frac{\beta c(I_1(t)+\phi_1 I_2(t)+\phi_2 I_3(t))}{N(t)}$ with $\beta$ being the probability of getting infected per sexual contact; $c$ is the effective contact rate; $\phi_1 \epsilon (0, 1)$ models the effect of a positive behavioural change as a result of knowing one's HIV positive status while $\phi_2 > 1$ accounts for increase in risky behaviour as a result of knowing one's HIV positive status. After infection with HIV, susceptible individuals infected with HIV will move into the class of HIV infected people not knowing their status ($I_1$). Individuals in the class ($I_1$) will know their HIV status at a rate *delta* through testing and counseling. A proportion $f$ of HIV positive people knowing their status will move into the class $I_2$ and the complementary $(1-f)$ will move into the class $I_3$ , respectively. HIV positive individuals who know their status will move into the sexually inactive class $I_4$ at a rate $\theta$. For simplicity, they assume the same $\theta$ value in both $I_2$ and $I_3$ classes. HIV positive people in classes $I_1, I_2, I_3$ and $I_4$ progress to the AIDS class ($A$) at a rate $\rho$. In all classes individuals experience natural death at a constant rate $\mu$ which is proportional to the number in each class. Individuals in the AIDS class have an additional disease-induced death rate $\nu$. Using the above information they developed flow diagram and the dynamical system. They obtained the reproduction and computed some relevant sensitivity indices of the reproduction number which measures initial disease transmission in order to quantify their impact on the disease dynamics. They also evaluated the impact of the increased fraction of sexually inactive HIV positive individuals on the HIV epidemic. The elasticity of $R_0$ with respect to $\theta$ which measures the effect a change in $\theta$ has as a proportional change in $R_0$. It is therefore evident that the abstinence rate $\theta$ (voluntary withdrawal from (risky) sexual behavior of HIV positive individuals) has a significant impact on the disease transmission. The authors suggested in their research that effective counseling and testing will be able to control the HIV/AIDS epidemic. They also suggested that giving free antiretroviral drugs to HIV positive individuals who change their sexual behavior and have withdrawn from sexual contacts may be an effective tool to control the epidemic.

A simple deterministic HIV/AIDS model incorporating condom use, sexual partner acquisition, behavior change and treatment as HIV/AIDS control strategies has been formulated by Nyabadza et al. in [33], the sexually active population at any time $(t)$ is stratified into those that are screened through HCT and those that are not. The population is divided into seven compartments of those that are susceptible and not screened $S_n(t)$, susceptible and screened $S_s(t)$, infected and unscreened $I_n(t)$, infected and screened $I_s(t)$, under treatment $I_T(t)$, who would have developed AIDS and not screened $A_n(t)$, and those who would have developed AIDS but having been screened $A_s(t)$. Using their assumption they developed the flow diagram that describe population movements between compartments and dynamical system. They calculated the sensitivity indices of $R_0$, for each model parameter and obtained the most sensitive parameter.

In [54], a population of size $N(t)$ at time $t$ with constant inflow of susceptible at a rate $Q$ is studied. The population size is divided into four subclasses, susceptible $(S)$, infective $(I)$ also assumed to be infectious, both pre-AIDS patients $(P)$ and AIDS patients $(A)$ are assumed to be sexually inactive, and therefore non-infectious. The natural mortality rate is $\nu$ in all classes and the disease induced death rate is $\alpha$ in the AIDS patients class. In addition, $\beta$ is a sexual contact rate, $c$ is the number of partners per individual and $\mu$ is the rate of movement of pre-AIDS class individuals into AIDS class. It is also assumed that the susceptible become HIV infected via sexual contacts with infective which may also lead to the birth of infected children. It is assumed that a fraction of new born children are infected at birth, and hence are directly recruited into the infective class with a rate $(1-\epsilon)\theta$ and others die effectively at birth $(0 \leq \epsilon \leq 1)$. Not only vertical transmission is considered for direct recruitment of infected persons within the population, but also infected immigrants are recruited directly into both infective and pre-AIDS patients classes. Consequently, $m(1-\pi)I$ is the recruitment rate of infective immigrants into the population and $m\pi I$ is the recruitment rate of pre-AIDS immigrants into the population. It is also assumed that some of the infective move to join pre-AIDS class, depending on the viral counts, with a rate $\sigma\delta$ and others with serious infection directly join the AIDS class with a rate $(1-\sigma)\delta$, where $(0 \leq \sigma \leq 1)$. The interaction between susceptible and infective is assumed to be of standard mass action type.Based on the above assumption they developed flow chart and dynamical system.Then they computed reproduction number

and analyzed their result.

HIV is one of the major life-threatening viruses that are spreading in the People's Republic of China. A susceptible exposed in the latent stage-infectious ($SEI$) model was established to sketch the evolution of epidemic. They divided the total population at time t into three mutually exclusive classes: susceptible individuals $S(t)$ and undiagnosed individuals $E(t)$ in the latent period and infected individuals $I(t)$ who have been diagnosed. Hence, $N(t) = S(t)+E(t)+I(t)$. Based on their assumption they developed flow diagram and ordinary differential equations for transmission dynamics of HIV/AIDS. By constructing Lyapunov function, globally asymptotical stabilities of the disease-free and endemic equilibria were given. Through the HIV/AIDS data in China, all parameters involved in $SEI$ model were analyzed and parts of them were estimated [118].

In order to find out the effect of human (sexual) behavior change and immigration in spreading the HIV/AIDS, a deterministic model of HIV/AIDS with infective immigration is formulated [72]. They divided the sexually active population $N(t)$ into six compartments, namely, susceptible individuals $S(t)$, infected individuals who unaware of their HIV/AIDS infection $I_1(t)$, infected individuals who aware of their HIV/AIDS infection $I_2(t)$, infected individuals who receive treatment $T(t)$, full-blown AIDS group $A(t)$ and removed class $R(t)$ at any time $t$. The model is formulated based on following assumptions.

• Since their purpose in this model is to see what effect the human behavior (including movement, sexual habits) can play in the dynamics of HIV/AIDS disease, they avoid to consider detailed clinical stages of HIV/AIDS infection, instead they classed the population in two ways, uninfected and infected group. Uninfected group divided into two different compartments according to their behavior towards safe sex. Infected individuals divided four different compartments according to whether the infected individual aware of his/her HIV infection status, whether the infected individual received treatment and whether the infected individual has developed the last stage of the disease, the full blown AIDS.

• Susceptible individuals are assumed to get infected by sexual contact with both aware and unaware infected individuals with different transmission rates. The assumption that aware infected individuals also take part in the transmission process is based on the fact that some aware infected individuals may practice low-efficiency safe sex measures (and a few of them

may transmit the disease intentionally), and susceptible individuals may not be aware of the infected situation of his/her partner, which make them more vulnerable to the disease. So the new generated infected individuals by aware infective individuals are assumed to be not aware of his/her infection at first and go to the unaware infected individuals class.

• The simplest conceptual framework based on homogeneous behavior gives us clear insights into how community based chemotherapy can influence epidemiological pattern and transmission success. Here the mixing of susceptible with infective is considered to be homogeneous and accordingly the incidence rate is assumed to be bilinear.

• All new born are susceptible, i.e., in our model vertical transmission do not account for.

• We assumed that individuals in the treatment class not only to receive the ART, but also to be served with knowledge about the HIV/AIDS disease so that they were persuaded to avoid unsafe sexual behaviors. Full-blown AIDS individuals are assumed too ill to sexually active, so the susceptible do not get infected through sexual contacts with individuals from these two groups.

• Inclusion of compartment $R$: It is true that an appreciable number of people are now changing their sexual habits sufficiently due to the awareness of the widespread nature of disease in society, the monumental deaths resulting from the disease, increasing knowledge of the agony and psychological trauma experienced by the infected individuals, and better enlightenment due to intense HIV/AIDS educational campaigns. Based on the above assumption they developed flow diagram and dynamical system.They obtained basic reproduction number. The geometrical approach is used to obtain the global asymptotic stability of endemic equilibrium. Their numerical findings were illustrated through simulations using MATLAB, which shows reliability of their model from the practical point of view.

The work done by [104], the mathematical modeling of the spread of HIV / AIDS disease among the population requires the whole human population to be divided in to four classes. The whole of the human population at any time t is a variable and is denoted by $N(t)$. The four classes are as follows: (i) susceptible class the population size of this class at any time t is denoted by $S(t)$. The susceptible human has not yet infected by the disease but likely to get infected in future. (ii) Unaware infective class the population size of this class at any time $t$ is denoted by $I_1(t)$. The unaware infective humans have already infected by the disease but they do not know that they were already infected. (iii) Aware infective class the population size of this class at any time t is denoted by $I_2(t)$. The aware infective humans have already

infected by the disease and they know that they were already infected and (iv) AIDS class the population size of this class at any time t is denoted by $A(t)$. The AIDS class people are already AIDS patients. They assumed that the people are recruited into susceptible class at a constant rate of $Q_0$. This recruitment into the susceptible class is due to natural births. The people of susceptible class are likely to become infected through sexual contact with the people of $I_1(t)$ and $I_2(t)$ classes. Thus, people from $S(t)$ will go to $I_1(t)$ with a rate of$[(\beta_1 I_1 + \beta_2 I_2)(\frac{S}{N})]$. Here the parameters $\beta_1$ and $\beta_2$ are the probabilities per one contact with which the disease transmits to susceptible people by unaware and aware infective humans respectively. In this model they considered $\beta_1 > \beta_2$. That is, the probability of transferring the disease to susceptible population by unaware infected person is more than by aware infected person. People of $S(t)$ after getting infected will initially go to $I_1(t)$ but not to $I_2(t)$. This is because, all the infected people are assumed to be initially unaware of the infection. Then based on their assumption they developed flow diagram and a system of nonlinear ordinary differential equations. By analyzing the system they obtained the reproduction number $R_0$.

In [10], they considered four compartments of the relevant population: the susceptible population, $S(t)$, the undiagnosed HIV-infected population, $X(t)$, the diagnosed HIV-infected population, $Y(t)$, and people diagnosed with AIDS, $Z(t)$. They assumed that the population can be partitioned into homogeneous sub population or compartments such that all individuals in a given compartment have the same intrinsic epidemiological properties. Their model reflected, members of the susceptible population are transmitted to the undiagnosed HIV-infected population by a rate of $\alpha$, decreased by natural death rate $\mu$, move to the diagnosed with AIDS class at a rate of $\beta$ and increased by constant recruitment rate of $\lambda$. Members of the undiagnosed HIV-infected population infected by sexual transmission move to the diagnosed class in two ways: one is through contact tracing a rate of $\alpha$ and the other is through a linear term that represents random or voluntary testing a rate of $\widehat{k}$ and decreased by natural death rate $\mu$ and move to the diagnosed with AIDS class at a rate of $\beta$. The diagnosed with AIDS class decreased by mortality rate of the population with AIDS $\widehat{\mu}$. In addition, they assumed that transmission via the diagnosed HIV-infected population and the AIDS population are negligible because of Cuba's extraordinarily successful health care system. In regard to the transmission dynamics in Cuba, 99% of the infections are through sexual intercourse and so, they neglected infection by nonsexual transmission. From the given assumption they presented schematic diagram and dynamical system. Finally they analyzed the system and put their results.

The transmission of Human Immunodeficiency Virus (HIV) that causes the Acquired Immuno Deficiency Syndrome (AIDS) is strongly associated with unprotected sex and at the present understanding this epidemic can reach higher prevalence threshold level when there are extensive sexual contacts between the sex workers and general population. In [80], the authors investigated a nonlinear model for studying the transmission dynamics of HIV/AIDS epidemic with emphasis on the role of female sex workers. Here, they considered only the heterosexual transmissions of HIV/AIDS and formulate the mathematical model by dividing the total adult population under consideration into three different classes: male, female and female sex workers. They assumed different rates of recruitment for different classes of the population. The equilibria of the model and their stability are discussed in detail. The basic reproduction number $R_0$ of the model is computed and it is shown that the disease-free equilibrium is stable only when $R_0 < 1$. When the associated reproduction number $R_0 > 1$, the endemic equilibrium is globally stable. Finally, the numerical simulations are reported to support the presented analytical results.

In reference [70], they developed the following assumptions: Susceptible children increased by rate of birth $b_1$, decreased by mortality rate $g$ and decreased by the proportion of babies born with HIV from HIV infected mothers by the rate $b_1\nu$ and the proportion of uninfected children who survive the developmental stage of 0 up to $a$ by the rate $e^{-(ga)}b_1$. Infected children increased by the proportion of babies born with HIV from HIV infected mothers by the rate $b_1\nu$, decreased by natural child mortality rate $g$ and infected children progress to AIDS by the rate $m$. The AIDS Cases of children increased by the rate $m$ from infected children and decreased by natural child mortality rate $g$ and disease related death rate $d$. The susceptible adults increased by the rate $e^{-(ga)}b_1$ and decreased by the natural death rate $\mu$, proportion of vaccinated $\alpha$ and sexual interaction with infected adult class and infected adult class who receive treatment at the rate of $\beta_1 c_1$ and $\beta_2 c_2$ respectively where $\beta_1$ is the per partnership transmission probability of a normal infective who is not treated, $\beta_2$ is the per partnership transmission probability of an infective who is treated and counseled. $c_1$ is the average number of new sexual partners acquired per unit time by those infected but not yet counseled and treated and $c_2$ is the average number of new sexual partners acquired per unit time by those counseled and treated. The vaccinated adults increased by $(\alpha - \delta)$ proportion of susceptible,

decreased by $(1-\theta)$ proportion of sexual interaction with infected adult class and infected adult class who receive treatment at the rate of $\beta_1 c_1$ and $\beta_2 c_2$ respectively and decreased by the natural death rate $\mu$. The removed adults increased by $\delta$ proportion of susceptible and decreased by the natural death rate $\mu$. The infective adults increased by sexual interaction of susceptible adults with infected adult class and infected adult class who receive treatment at the rate of $\beta_1 c_1$ and $\beta_2 c_2$ respectively and $(1-\theta)$ proportion of sexual interaction of vaccinated adults with infected adult class and infected adult class who receive treatment at the rate of $\beta_1 c_1$ and $\beta_2 c_2$ respectively and decreased by proportion of the infective receiving treatment $\epsilon$, the rate at which the infective who do not receive treatment progress to HIV/AIDS $\eta$ and the natural death rate $\mu$. The infected adults who receive treatment increased by proportion of the infective receiving treatment $\epsilon$ and decreased by the rate at which those treated progress to AIDS $\lambda$ and the natural death rate $\mu$. The number of full blown AIDS Cases in adults increased by the rate at which the infective who do not receive treatment progress to HIV/AIDS $\eta$ and the rate at which those treated progress to AIDS $\lambda$, and decreased by the natural death rate $\mu$ and disease related death rate $d$. The transmission of HIV from an infective to a susceptible is through heterosexual mode and vertical transmission, there is random mixing of individuals within the population, AIDS cases has full blown symptoms and are easily noticeable and are not sexually interacted with and as such, they don't transmit the virus and do not give birth to new born, individuals in group I comprise of sexually Immature children aged 0 up to $a$ years and therefore do not transmit the disease, the removed class are sexually interacted with but are not infectious and are immuned, treatment is done in the adult group only, the vaccine acts both as the " Leaky type " and the " All or Nothing type " of vaccine. Using the above assumptions they developed the flowchart and dynamical systems. They analyzed the stability of the Disease Free Equilibrium, in group II since it is this group that is sexually active and responsible for the spread. They also assumed that the AIDS cases $A(t)$ in the population can easily be identified from the full blown symptoms and are not associated with sexually and as such are not involved in the spread of the diseases. They used the proportions of the populations to enable them study the steady states.

In [5], a nonlinear mathematical model is proposed and analyzed to study dynamics of HIV/AIDS with treatment and vertical transmission. In modeling the dynamics, the population of size $N(t)$ at time $t$ with constant inflow of susceptible with rate $\pi N$ where $\pi$ is the rate of recruitment into susceptible population is divided into five groups: Susceptible $S(t)$, infective $I(t)$,

(also assumed to be infectious), pre-AIDS patients $P(t)$, treated class $T(t)$, and AIDS patients $A(t)$ with natural mortality rate $\mu$ in all classes. They assumed that: the susceptible become HIV infected via sexual contacts with infective which may also lead to the birth of infected children. A fraction of new born children are infected during birth and hence are directly recruited into the infective class with a rate $(1-\epsilon)\theta$ and others die effectively at birth $(0 \leq \epsilon \leq 1)$ where $\epsilon$ is the fraction of newborns infected with HIV who dies immediately after birth and $\theta$ is the rate of newborns infected with HIV. They did not consider direct recruitment of the infected persons but by vertical transmission only. It is also assumed that some of the infective join the pre-AIDS class, depending on the viral counts, with a rate $\sigma_1\delta$ where $\delta$ is the rate of movement from infectious class and $\sigma_1$ is the fraction of $\delta$ joining the pre-AIDS class. They then proceed with a rate $\gamma$ to develop full blown AIDS. Some of the infective proceed to join the treated class with a rate $\sigma_2\delta$ where $\sigma_2$ is the fraction of $\delta$ joining treated class and then proceed with a rate $k$ to develop full blown AIDS while others with serious infection directly join the AIDS class with a rate $(1-\sigma_1-\sigma_2)\delta$. A Fraction of $\nu$ is assumed to get treatment. To simplify the model they assumed that the AIDS patients and those in pre-AIDS class are isolated and sexually inactive and hence they are not capable of producing children and also they do not contribute to viral transmission horizontally and are negligible. Taking into account the above considerations, they prepared the schematic flow diagram and dynamical system. Analysis of the model allows to determine the impact of treatment and vertical transmission on the transmission of HIV/AIDS infection in a population and found the basic reproduction number. A numerical study of the model has been conducted to see the effect of certain key parameters on the spread of the disease. It is shown that by controlling the rate of vertical transmission, the spread of the disease can be reduced significantly.

[95] They proposed a simple HIV/AIDS model which incorporates time delay during which a newly born infected child attains sexual maturity and becomes infectious. In this model, the sexually mature population is divided into three subclasses: the susceptible, the infective (also assumed to be infectious) and the AIDS population whose numbers are denoted by $S, I$ and $A$. The number of total population is denoted by $N(t)$, at any time $t$. In the model, they assumed that the susceptible become HIV infected via sexual contacts with infective. It was also assumed that all newborns are infected at birth. They did not consider direct inflow of other infected persons except through vertical transmission as their purpose was to study the role of delay, modeled as a period of sexual maturity of infected newborns. In the model, they have

assumed that a fraction of infected newborns, who sustain treatment, joins the infective class while the others, who do not sustain treatment, joins AIDS class after getting sexual maturity. The infective through vertical transmission at any time $t$ is given by $\gamma\epsilon I(t-\tau)$, because those infected at time $(t-\tau)$ becomes infectious at time $\tau$ later, if they do not develop AIDS by that time. The fraction of infective which develop AIDS during the period of getting sexual maturity, if they survive the maturity period joins the AIDS class. However, for the model to be biologically reasonable, it may be more realistic to assume that not all those infected will survive after time $\tau$ units, and this claim supports the introduction of the survival term $e^{-d\tau}$. Thus, in their model the term $\gamma\epsilon I(t-\tau)e^{-d\tau}$ represents the introduction of infective persons who survive the maturity period $\tau$ in which the time taken to become infectious is $\tau$. Here $e^{-d\tau}$ represents the probability that an individual survives the maturity period $[t-\tau, t]$ such that $0 < e^{-d\tau} \leq 1$. It is also assumed that some of the infective move to AIDS class with a rate coefficient $\delta$ to develop full blown AIDS. Using the above assumptions they constructed schematic diagram and developed system of nonlinear ordinary differential equations. From the dynamical system they obtained the two equilibrium points namely infection-free equilibrium and the endemic equilibrium and also they found the basic reproduction number and they did the analysis of the two equilibrium points.

[68] To construct the model, they first divided the total population into a susceptible class of size $S$ and an infectious class before the onset of AIDS and a full-blown AIDS group of size $A$ which is removed from the active population. Based on the facts that the infectious period is very long ($\geq 10 years$), they further considered several stages of the infectious period. For simplicity, they only considered two stages, the asymptomatic phase ($I$) and the symptomatic phase ($J$). By all sorts of treatment methods, some individuals with the symptomatic phases can be transformed into asymptomatic individuals. By introducing discrete time delay (onset of treatment effects) to the model, they shall establish the delay differential equation model. The model has schematic representation and they established dynamical system. Investigating the systems, they obtained the basic reproduction number by the method of next generation matrix. They also found a disease-free equilibrium and the endemic equilibrium and analyzed their stability.

[92] Consider a population of size $N(t)$ at time $t$ with constant inflow of susceptible at a rate $Q_1$

and that of HIV infective at a rate $Q_2$ into the population. The population size $N(t)$ is divided into four subclasses of susceptible $S(t)$, infective $I(t)$ (also assumed to be infectious), pre-AIDS patients $P(t)$ and that of AIDS patients $A(t)$ with natural mortality rate $d$ in all the classes. The susceptible become infected via sexual contacts with infective and with those in pre-AIDS class. The total population $N$ looses individuals at a higher rate from the class $A$ than the others. It may be noted that the individuals in pre-AIDS class may also interact sexually owing to illiteracy, ignorance or other social factors especially in underdeveloped nations but the contact rate may be very less in comparison to that of other infective ($\beta^{'} \ll \beta$) . It is also assumed that a fraction $\epsilon(0 \leq \epsilon \leq 1)$ of all infective i.e. $\epsilon\delta I$ goes to pre-AIDS class depending on the level of viral count) while the others with serious infection i.e. $(1-\epsilon)\delta I$ directly join the AIDS class. However, it is assumed that virtually all individuals in pre-AIDS class will ultimately develop the disease to join AIDS class. Here $\delta$ is the rate of movement from infectious class, so that $\frac{1}{\delta}$ denotes the average incubation period,$\beta^{'}$ and $\beta$ are the contact rates of susceptible with infective and pre-AIDS individuals respectively. With the above assumptions and considerations they presented the flowchart and dynamical system. They analyzed the existence and stability of the equilibrium points of the model systems. They showed the systems do not exhibit a disease-free equilibrium due to direct inflow of infective at a constant rate. However, there exists only one non-negative equilibrium point of the model. This endemic equilibrium $E^*$ exists when HIV infection persists in the population. They also considered model without inflow of HIV infective including interaction with pre- AIDS, then the model exhibits two non-negative equilibria namely the disease-free equilibrium and the endemic equilibrium. The disease- free equilibrium $E_0$ used to define the basic reproduction number $R_0$ and they checked the stability of $E_0$ using $R_0$.

Reference [93], proposed and analyzed a nonlinear mathematical model to study the effect of vaccination on the spread of HIV/AIDS in a homogeneously mixing population of variable size has been studied qualitatively using stability theory of nonlinear differential equations. Here the total population is divided in to four disjoint groups: Susceptible $S(t)$, vaccinated $V(t)$, infective $I(t)$ and AIDS patients $A(t)$. The model assumptions are:

- the susceptible become infected via sexual contacts with infective $I(t)$ and new infections are generated at a rate $\beta_1$ with constant immigration rate $Q_0$ and natural mortality rate $d$. The susceptible population is vaccinated at a constant rate $\Phi$.
- the vaccinated population is generated when susceptible population gets vaccinated at a con-

stant rate $\Phi$. The vaccine has the effect of reducing (but does not eliminate) the infection rate by a factor of $\sigma$. It is diminished by natural deaths at a rate $d$ and by the vaccination wears off at a rate $\theta$.

- infected population is generated by the HIV infection of susceptible and some fraction of vaccinated individuals by sexual contacts with infective. It is diminished by natural mortality rate $d$ and by the development of clinical AIDS at a rate $\delta$.

- The population of individuals with clinical AIDS, $A(t)$, generates when infective population $I(t)$ looses individuals with disease symptoms at a rate $\delta$. This population suffers by natural mortality at a rate $d$ and by disease induced deaths at a rate $\alpha$. Then they represented the schematic diagram and developed dynamical system. The model incorporates three important parameters, the vaccination campaign $\Phi$, measure of vaccine efficiency $\sigma$ and the rate at which vaccine wears off $\theta$. It is found that a vaccination campaign $\Phi$, howsoever large, may fail to eradicate the disease. However, if the vaccination strategy is such that $R(\Phi) < 1 < R_0$ (i.e. $\sigma$ is low enough) then without vaccination HIV infection will persist in the population and increasing the rate of vaccination would lead to ensure disease eradication.

In [94] they considered a population of size $N(t)$ at time $t$ with constant inflow of susceptible with a rate $Q_0$. The population size $N(t)$ is divided into four subclasses of susceptible $S(t)$, infective $I(t)$ (also assumed to be infectious), pre-AIDS patients $P(t)$ and AIDS patients $A(t)$ with natural mortality rate $d$ in all the classes. In the model, they assume that the susceptible become HIV infected via sexual contacts with infective which may also lead to the birth of infected children. It is assumed that a fraction of new born children are infected at birth and hence are directly recruited into the infective class with a rate $(1-\epsilon)\theta$ and others die effectively at birth ($0 \leq \epsilon \leq 1$). They did not consider direct recruitment of other infected persons but by vertical transmission only. The interaction between susceptible and infective is assumed to be of standard mass action type. It is also assumed that some of the infective move to join pre-AIDS class,depending on the viral counts, with a rate $\sigma\delta$ and then proceed with a rate $\mu$ to develop full blown AIDS while others with serious infection directly join the AIDS class with a rate $(1-\sigma)\delta$ where $0 \leq \sigma \leq 1$. To simplify the model, it is reasonable to assume that the AIDS patients and those in pre-AIDS class are exposed and sexually inactive as they are isolated and hence are not capable of producing children they also do not contribute to viral transmission horizontally are taken negligible. It is remarked here that these assumptions are valid in developed countries following stringent screening measures but may not be true

in under developed nations due to poor medical facilities or the social stigma attached with the disease. Using the above assumptions they constructed flowchart and developed dynamical system. They analyzed the model and obtained the result that in order to reduce the spread of the disease, the number of sexual partners as well as unsafe sexual interaction with an infective is to be restricted. They also found that the disease becomes more endemic due to immigration. If the rate of migration is restricted into susceptible community, the spread of the disease can also be kept under control. The effect of an increase in disease-induced death rate is, however, to decrease the AIDS patients population.

A model for the HIV-infection transmission in a male homosexual cohort analyzed by considering two types of infected individuals. Those that are infected but not under any sort of clinical or therapeutical treatment and those who are under treatment. The two groups of infective differ in their incubation time, contacts with the susceptible individuals, and probability of transmission. The analytical results show that change in sexual behavior is important in lowering prevalence and incidence rate and, eventually, in driving the population toward the disease-free equilibrium [115].

Mathematical models here serve as tools for understanding the epidemiology of Human Immunodeficiency Virus (HIV) and Acquired Immunodeficiency Syndrome (AIDS) if they are carefully constructed. The research emphasis is on the epidemiological impacts of AIDS and the rate of spread of HIV/AIDS in any given population through the numericalization of the Basic reproductive rate of infection ($R_0$). Applicable Deterministic models, Classic Endemic Model (SIR), Commercial Sex Workers (CSW) model, Dynamic model and Stability Analysis are explained. The models show that AIDS disease progressively increases with years and it is thus concluded that if the current trend is unchecked, a catastrophic AIDS epidemic (Pandemic) will occur in the near future [82].

In [20], they proposed and analyzed a mathematical model for HIV/AIDS transmission with varying population size in a homogeneously mixing population. The model subdivides the human population into four mutually-exclusive compartments: susceptible individuals ($S$); HIV-infected individuals with no clinical symptoms of AIDS (the virus is living or developing in the individuals but without producing symptoms or only mild ones) but able to transmit

HIV to other individuals ($I$); HIV-infected individuals under ART treatment (the so called chronic stage) with a viral load remaining low ($C$); and HIV-infected individuals with AIDS clinical symptoms ($A$). They assumed that HIV-infected individuals with and without AIDS symptoms, have access to ART treatment. HIV-infected individuals with no AIDS symptoms,I, progress to the class of individuals with HIV infection under ART treatment, $C$, at a rate $\phi$, and HIV-infected individuals with AIDS symptoms are treated for HIV at a rate $\alpha$. They also assumed that HIV-infected individuals with AIDS symptoms, $A$, that start treatment, move to the class of HIV-infected individuals, $I$, and will move to the chronic class, $C$, only if the treatment is maintained. HIV-infected individuals with no AIDS symptoms, $I$, that do not take ART treatment, progress to the AIDS class, $A$, at rate $\rho$. Only HIV-infected individuals with AIDS symptoms, $A$, suffer from an AIDS induced death, at a rate $d$. These assumptions represented by the flow chart and dynamical system. They analyzed the dynamics and obtained the two equilibria.

Based on the above Literature review we developed the Mathematical model on the dynamics of Human Immunodeficiency Virus/Acquired Immunodeficiency Syndrome (HIV/AIDS) infection in Ethiopia.

# Chapter 3

# Methodology

## 3.1 Data analysis

Mathematical modeling of complex biological systems can be mostly carried out in a deterministic manner. Deterministic models of epidemiology are usually described by differential equations. Of which are two types; ordinary differential equations and partial differential equations. In this thesis, our models derived from ordinary differential equations. They were analyzed by classifying their steady states. We now define and give theorems that are relevant to the thesis.

### 3.1.1 Ordinary differential equations

***Definition:*** (Ordinary differential equations)
An ordinary differential equation of order $n$ is an equation which contains derivatives of an unknown function $y(t)$ which is denoted as $f(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \frac{d^3y}{dt^3}, ..., \frac{d^ny}{dt^n}, t) = 0$
***Definition:*** (System of ordinary differential equations)
A system of differential equations is a system which contains two or more number of differential equations at the same time which is denoted as

$$\frac{dx_i}{dt} = F(x(t), t)$$

Where

$$x(t) = (x_1(t), x_2(t), x_3(t), ..., x_n(t))^T$$

$$F = (f_1, f_2, f_3, ..., f_n)^T \quad and \quad f_i = f_i(x_1(t), x_2(t), x_3(t), ..., x_n(t), t)$$

***Definition:*** (steady states / equilibrium points)

The point $X_*$ in $R^n$ is an equilibrium point or steady state of the system of the first order differential equation $\frac{dX}{dt} = F(X)$ is obtained by making $\frac{dX}{dt} = 0$ and satisfies $F(X_*) = 0$.

### 3.1.2 Stability of the steady states

Stability by linearization

For most dynamical systems the equilibrium point (fixed point) of a system of nonlinear differential equations plays an important role in the analysis of the models, we therefore give the definition of a fixed point and describe the analysis of the fixed point below. Let $f : R^n \longrightarrow R^n$ be a $C^1$ map and suppose that $p$ is a point such that $f(p) = 0$. That is $p$ is an equilibrium point for the differential equation $y^{'}(t) = f(y(t))$. The linear part of $f$ at $p$ denoted by $Df(p)$ is the matrix of partial derivative at $p$. For $y \epsilon R^n$ we have

$$f(y) = \begin{pmatrix} f_1(y) \\ f_2(y) \\ . \\ . \\ . \\ f_n(y) \end{pmatrix}$$

The functions $f_i$ are called the components of $f$. We define

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(p) & \frac{\partial f_1}{\partial y_2}(p) & \dots & \frac{\partial f_1}{\partial y_n}(p) \\ \frac{\partial f_2}{\partial y_1}(p) & \frac{\partial f_2}{\partial y_2}(p) & \dots & \frac{\partial f_2}{\partial y_n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}(p) & \frac{\partial f_n}{\partial y_2}(p) & \dots & \frac{\partial f_n}{\partial y_n}(p) \end{pmatrix}$$

Called Jacobian matrix and the stability of a flow of a nonlinear system can be studied using different approaches if this algebraic sign of the Eigenvalues of the Jacobian matrix is easily identified. But if the algebraic sign of these Eigenvalues are not determined easily we can use and we restrict our self to Routh-Hurwitz stability criterion.

### 3.1.3 Routh – Hurwitz stability criterion

Finding the roots of the characteristic equation of the Jacobian matrix i.e. poles to determine the stability of the system. In the analysis, the characteristic equations are mostly large and complex. Hence it is difficult to simplify them into roots (like those of order 4 and above). For such situations, Routh- Hurwitz Method provides an easy and quick method to determine the stability without the need to disintegrate the characteristic equation. Routh- Hurwitz Stability Criterion is based on ordering the coefficients of the characteristic equation into an array, also known as Routh Array.

Suppose the characteristic equation of a Jacobian matrix of the system is given as:

$$p_n(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_1\lambda^1 + a_0 = 0$$

To determine whether this system is stable or not, we have to check the two necessary but not sufficient conditions that all the roots have negative real parts:

(a) all the polynomial coefficients must be the same sign and

(b) all the polynomial coefficients must be nonzero.

Thus; if these condition are satisfied, then from the given equation, we will form Routh Array as shown below:

| | | | | |
|---|---|---|---|---|
| $\lambda^n$ | $a_n$ | $a_{n-2}$ | $a_{n-4}$ | $a_{n-6}...$ |
| $\lambda^{n-1}$ | $a_{n-1}$ | $a_{n-3}$ | $a_{n-5}$ | $a_{n-7}...$ |
| $\lambda^{n-2}$ | $b_1$ | $b_2$ | $b_3$ | $b_4...$ |
| $\lambda^{n-3}$ | $c_1$ | $c_2$ | $c_3$ | $c_4...$ |
| . | . | | | |
| . | . | | | |
| . | . | | | |
| $\lambda^1$ | | | | |
| $\lambda^0$ | | | | |

Where the $a_i^{'}s$ are the polynomial coefficients and the coefficients in the rest of the table are computed using the following pattern.

To determine whether this system is stable or not, check the following conditions:

1. Two necessary but not sufficient conditions that all the roots have negative real parts are: all the polynomial coefficients must have the same sign and all the polynomial coefficients must be nonzero. If this condition is satisfied then compute the Routh-Hurwitz array as follows

$$b_1 = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{-1}{a_{n-1}}(a_n a_{n-3} - a_{n-2}a_{n-1})$$

$$b_2 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = \frac{-1}{a_{n-1}}(a_n a_{n-5} - a_{n-4}a_{n-1})$$

$$b_3 = \frac{-1}{a_{n-3}} \begin{vmatrix} a_{n-2} & a_{n-6} \\ a_{n-3} & a_{n-7} \end{vmatrix} = \frac{-1}{a_{n-3}}(a_{n-2}a_{n-7} - a_{n-3}a_{n-6})$$

$$c_1 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} = \frac{-1}{b_1}(a_{n-1}b_2 - b_1 a_{n-3})$$

$$c_2 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} = \frac{-1}{b_1}(a_{n-1}b_3 - b_1 a_{n-5})$$

2. The necessary condition that all roots have negative real parts is that all elements of the first column of the array have the same sign. The number of changes of sign equals the number of roots with positive real parts.

3. Special case 1: The first element of a row is zero, but some other elements in the row are nonzero. In this case, simply replace the zero elements by $\epsilon$, complete the table development, and then interpret the results assuming that $\epsilon$ is a small number of the same sing as the element above it. The results must be interpreted in the limit as $\epsilon \to 0$.

4. Special case 2: All the elements of a particular row are zero. In this case, some of the roots of the polynomial are located symmetrically about the origin of the plane. e.g: a pair of purely imaginary roots. The zero row will always occur in a row associated with an odd power of $\lambda$. The row just above the zero row holds the coefficients of the auxiliary polynomial. The roots of the auxiliary polynomial are the symmetrically placed roots. Be careful to remember that the coefficients in the array skip powers of $\lambda$ from one coefficient to the next.

### 3.1.4 Basic Reproduction Number

Two important concepts in modeling outbreaks of infectious diseases are the basic reproduction number, universally denoted by, and the generation time (the average time from symptom onset in a primary case to symptom onset in a secondary case), which jointly determine the likelihood and speed of epidemic outbreaks.

In studying any epidemiological model: Identifying the threshold value is extremely important. This threshold quantity which determines whether an epidemic occurs or the disease simply dies out. This quantity is called the basic reproduction number, denoted by $R_0$ which can be defined as the number of secondary infections caused by a single infective introduced into a population.

It is a concept in the epidemiology of infectious diseases and is a measure of how infectious a disease is, and is required if you wish to calculate how many people you need to vaccinate if you are to achieve. When somebody gets an infectious disease, they may pass it on to nobody else, or they may infect one, two or more other people, who become secondary cases. The reproduction number usually denoted by $R_0$, is the average or mean number of secondary cases caused by each case of an infectious disease, during the infectious period [67]. Basically the reproduction number will depend on a large number of factors, including: How the infectious organism is spread? Behaviors which affect the likelihood of spread such as social mixing, sexual and feeding practices and so on. The basic reproduction number $R_0$ is also known as basic reproductive rate or basic reproductive ratio is the expected number of secondary cases produced by a typical primary case in an entirely susceptible population. When $R_0 < 1$ the infection will die out but any value $R_0 > 1$ implies it will spread without control measures and higher numbers are more likely to cause epidemics. In cases where $R_0 = 1$, the disease becomes endemic, meaning the disease remains in the population at a consistent rate, as one infected individual transmits the disease to one susceptible. The basic reproduction number $R_0$ is proportionate to: the length of time that the case remains infectious (duration of infectiousness), the number of contacts a case has with susceptible hosts per unit time (the contact rate), and the chance of transmitting the infection during an encounter with a susceptible host (the transmission probability) and can be expressed mathematically as:

$$R_0 = cpd$$

Where $c$ the number of contacts per unit time,$p$ is the transmission probability per contact, and $d$ is the duration of infectiousness [44]. In other words this infective individual makes $\beta N$ contacts per unit time producing new infections with a mean infectious period of $\frac{1}{\gamma}$ and therefore, the basic reproduction number is obtained by

$$R_0 = \frac{\beta}{\gamma}$$

This value quantifies the transmission potential of a disease. If the basic reproduction number falls below one i.e. the infective may not pass the infection on during the infectious period, the infection dies out and if $R_0 > 1$ there is an epidemic in the population [[108], [109]].

**A method for calculating basic reproduction number**

Here we briefly sketch and apply to a more epidemiological model the method by Van den Driesch and Warmouth (2002) on calculating the basic reproduction number $R_0$. Consider

a heterogeneous population whose individuals are distinguishable by age, behaviour, spatial position and/or stage of the disease, but nevertheless can be grouped in to $n$ homogeneous compartments. That is, the parameters may vary compartment to compartment, but are identical for all individuals within a given compartment. Let $x = (x_1, ..., x_n)^T, x_i \geq 0$ for all $i = 1, ..., n$ are the vector of densities of individuals in each compartment. Let us sort the compartments so that the first $m$ compartments correspond to infected or addicted individuals.

In order to compute $R_0$, it is important to distinguish new infections from all other changes in the host population. Let $F_i(x)$ be the rate of appearance of new infections in compartment $i$, $V_i^+(x)$ be the transfer rate of individuals into compartment $i$ by all other means, and $V_i^-(x)$ be the rate transfer of individuals out of compartment $i$. It is assumed that each function is continuously differentiable at least twice in each variable. Any model of infectious disease dynamics can be formulated as follows:
$\frac{dx_i}{dt} = f_i(x) = F_i(x) - V_i(x)$where $i = 1, ..., n$
$V_i(x) = V_i^-(x) - V_i^+(x)$
If $x_0$ is the disease free equilibrium (DFE) and $f_i(x)$ satisfy those technical assumptions, then the derivatives $Df(x_0)$ and $DV(x_0)$ can be partitioned as

$$\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \quad and \quad \begin{bmatrix} v & 0 \\ J_3 & J_4 \end{bmatrix}$$

where $F$ and $v$ are the $m \times m$ matrices defined by

$$F = \left[\frac{\partial f_i}{\partial x_j}(x_0)\right] \quad and \quad v = \left[\frac{\partial v_i}{\partial x_j}(x_0)\right] \text{ with } 1 \leq i \ , \ j \leq m$$

Moreover, $f$ is non-negative, $v$ is invertible with eigenvalues whose real parts are positive, and all eigenvalues of $J_4$ have positive real part.

Now, consider the rate of infected or addicted individual introduced into compartment $k$ of a disease-free population. The $(j, k)$ entry of $v^{-1}$ is the mean length of time this individual spends in compartment $j$ during its life time, assuming that the population remains near the disease free equilibrium DFE and barring re-infection or re-addicted. The $(i, j)$ entry of $F$ is the rate at which infected individuals in compartment $j$ produce new infections in compartment $i$. Hence the $(i, j)$ entry of the product $FV^{-1}$ is the expected number of new infections in compartment $i$ produced by the infected or addicted individual originally introduced into compartment $k$.

The matrix $Fv^{-1}$ is usually referred to as the next generation matrix. Setting

$$R_0 = \rho(Fv^{-1})$$

where $\rho(Fv^{-1})$ denotes the spectral radius of a matrix $Fv^{-1}$ (that is, the Eigenvalue with the maximum absolute value). The theorem of van den Driesch and Warmouth (2002) states that, $R_0$ is a threshold parameter for local stability of the DFE; consider the disease transmission model given by $(A)$ with $f_i(x)$ satisfying the above mentioned technical assumptions. If $x_0$ is the DFE of this model, then $x_0$ is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

## 3.2 Sensitivity analysis

The parameter values and assumptions of any model are subject to change and error. Sensitivity analysis (SA) is the investigation of these potential changes & errors and their impacts on conclusions to be drawn from the model. Sensitivity analysis is commonly used to determine the robustness of model predictions to parameter values (since there are usually errors in data collection and presumed parameter values). Here we use it to discover parameters that have a high impact on reproduction number $R_0$, and should be targeted by intervention strategies.

We calculate the sensitivity indices of the reproductive number, $R_0$, to the parameters in the model. These indices tell us how crucial each parameter is to disease transmission and prevalence. The normalized forward sensitivity index of a variable to a parameter is the ratio of the relative change in the variable to the relative change in the parameter. When the variable is a differentiable function of the parameter, the sensitivity index may be alternatively defined using partial derivatives [78].

***Definition:*** The normalized forward sensitivity index of a variable, $u$ that depends differentiable on a parameter $p$, is defined as:

$$SI(p) = \frac{\partial u}{\partial p} \times \frac{p}{u}$$

If the magnitude of sensitivity index is high for the parameter $p$ out of other parameters then we say that $p$ is more sensitive parameter.

## 3.3 Liapunov function

Technique for verifying stability is to seek an aggregated summarizing function that continually decreases towards a minimum as the system evolves. Suppose that $\overline{x}$ is an equilibrium point of a given dynamical system. A Liapunov function for the system and the equilibrium point $\overline{x}$ is a real valued function $V$; which is defined over a region $\Omega$ of the state space that contains $\overline{x}$ and satisfies the three requirements:

1. $V$ is continuous
2. $V(x)$ has a unique minimum at $\overline{x}$ with respect to all other points in $\Omega$.
3. Along any trajectory of the system contained in $\Omega$; the value of $V$ never increases.

***Definition:*** A function $V$ defined on a region $\Omega$ of the state space and containing $\overline{x}$ is a Liapunov function if it satisfies the following three requirements

1. $V$ is continuous and has continuous first partial derivatives
2. $V(x)$ has a unique minimum at $\overline{x}$ with respect to all other points in $\Omega$
3. The function $V^{'}(x) = \nabla V(x) f(x)$ satisfies $V^{'}(x) \leq 0$ for all $x(t)$ in $\Omega$.

**Theorem 3.1.** *(Liapunov Theorem)*
*If there exists a Liapunov function $V(x)$; then the equilibrium point $\overline{x}$ is stable. If, furthermore, the function $V^{'}(x)$ is negative for every point then the stability is asymptotic.*

# Chapter 4

# Treatment and Inflow Infective Immigrants on the Dynamics of HIV/AIDS

## 4.1 Introduction

Diseases can be transmitted many ways, some of which can be classified as either horizontal or vertical. In the case of HIV/AIDS, horizontal transmission can result from direct physical contact between an infected individual and a susceptible individual. Vertical transmission, on the other hand, can result from direct transfer of a disease from an infected mother to an unborn or newborn offspring. Diseases that can be transmitted vertically include chagas, dengue fever, hepatitis B and HIV/AIDS just to name a few. Vertical transmission of HIV/AIDS can occur during pregnancy, delivery or breastfeeding and is influenced by many factors, including maternal viral load and the type of delivery [88]. According to [7] and [111], about 20% of the children infected with HIV develop AIDS in the first year of their lives, and most of them die by the age of 4 years. The others, up to 80% of infected children, develop symptoms of HIV/AIDS at school entry age (7-9 years) or even during adolescence.

The first simple HIV Mathematical epidemic model goes back to Anderson [91] in 1986. By then behaviour change was recognized as the major way of combating the spread of HIV/AIDS epidemic given that there was no treatment or vaccine to the virus. After the discovery of Anti-retroviral treatment, modeling of HIV/AIDS was directed towards incorporating behaviour

change and effects of treatment. Incorporating treatment and social behaviour posed a challenge to HIV/AIDS mathematical modeling because treatment acts both in the positive and negative direction. It reduces the infectiousness of an infected individual reducing the probability of transmission from an infective to a susceptible. On the contrary anti- retroviral therapies increases the lifespan of the HIV Infectives and as such they can infect more people if the treatment does not reduce infectiousness with no change in social behaviour. These directions included the models by Valesco - Hernandez and Hsieh 1994 [40] who concluded that only significant reductions in the transmission probability can contain the spread of the epidemic. Such reductions could be through adoption of safer sexual practices or through reductions in viral load due to treatment. A model by Ying -Yen and Cooke [41] in 2000 on Behaviour change and treatment of core groups and its effects on the spread of HIV/AIDS showed that behaviour change and treatment can eradicate the disease however if the treatment and behaviour change levels do not reach critical values, detrimental effects could be realized resulting from slower progression to AIDS without sufficiently lower transmission rates resulting in increased spread of HIV infection.

HIV/AIDS transmission in Africa is primarily through heterosexual sex and vertical transmission (mother-to-child). Forty percent of all HIV/AIDS cases result from mother to child transmission [35]. The impact of migration of population on the distribution and spread of HIV/AIDS disease has to be analyzed properly and must be understood clearly. Migration and immigration of the people from one country to another country due to different reasons play a crucial role in the evolution and spread of HIV/AIDS epidemic [22, 23, 35].

The study of HIV transmission and the dynamics of the disease have been of a great interest to both applied mathematicians and Biologists. Mathematical modeling has proved to be an important tool in analyzing the spread and control of HIV disease [9, 83]. The results of modeling and analysis help to improve understanding of the major contributing factors to the pandemic. Mathematical models have been studied and important inferences have been drawn in case of epidemics like Ebola, Breast cancer, Malaria, Tuberculosis and Influenza [6, 24, 25, 60, 92].

Several researchers have developed HIV/AIDS models so as to understand and explain the dynamics and the spread of the disease and succeeded to a large extent. Modeling and Analysis of the spread of AIDS epidemic with immigration of HIV infective is studied in [38, 92]. A theoretical framework describing the transmission of HIV/AIDS with screening of unaware infective persons is presented in [100, 101]. The joint effect of both medical screening and variable inflow of aware and unaware infective immigrants on the disease transmission has been studied by [55]. Modeling the Combined Effect of Vertical Transmission and Variable Inflow of

Infective Immigrants on the Dynamics of HIV/AIDS has been studied by [104]. The spread of the disease due vertical transmission has also been studied by [14]. In this paper, we proposed an improvement of the model [104] Modeling the Combined Effect of Vertical Transmission and Variable Inflow of Infective Immigrants on the Dynamics of HIV/AIDS. The model [104] forms the motivation for the present study. Here we have investigated the combined effect of unaware infective immigrants, different mode of transmissions and aware infective immigrants, on the dynamics of HIV/AIDS. The results are presented graphically and discussed qualitatively in the following sections.

## 4.2 Mathematical Model

Modeling the Combined Effect of Vertical Transmission and Variable Inflow of Infective Immigrants on the Dynamics of HIV/AIDS studied in [104]. The flow diagram of the model and the non- linear deterministic model of the problem are given as follows.

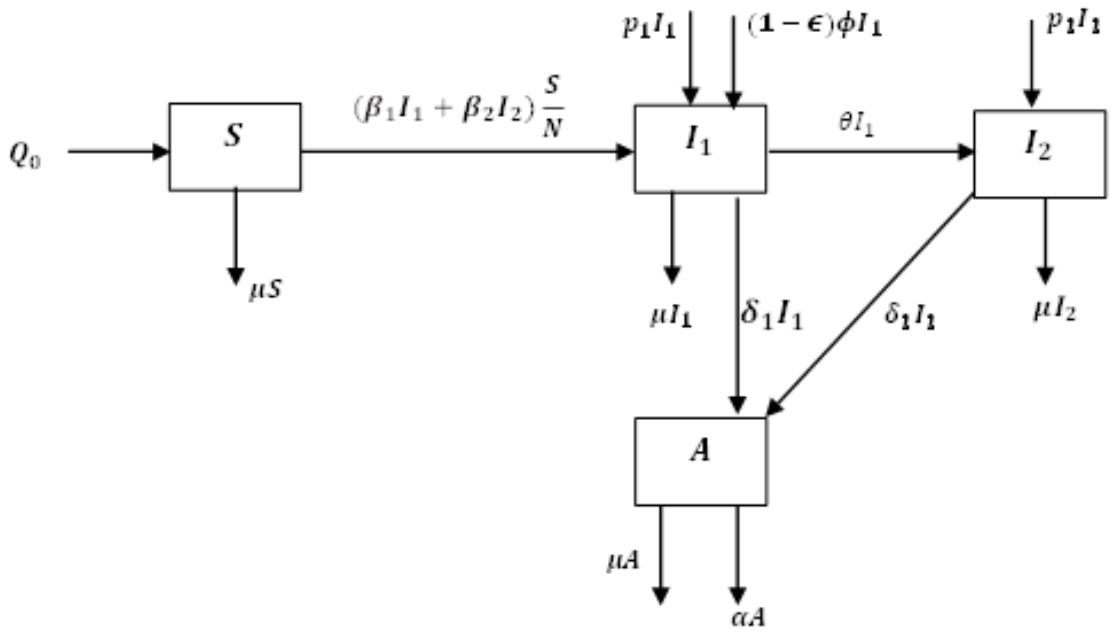


Figure 4.1: Flow diagram of the model [104].

The dynamical system of [104] is given as follows:

$$\frac{dS}{dt} = Q_0 - (\frac{\beta_1 I_1}{N} + \frac{\beta_2 I_2}{N})S - \mu S$$
$$\frac{dI_1}{dt} = (\frac{\beta_1 I_1}{N} + \frac{\beta_2 I_2}{N})S + p_1 I_1 + (1-\epsilon)\phi I_1 - (\theta + \delta_1 + \mu)I_1$$
$$\frac{dI_2}{dt} = p_2 I_2 + \theta I_1 - (\delta_2 + \mu)I_2$$
$$\frac{dA}{dt} = \delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu)A$$

Here the initial conditions are considered to be

$$S(0) = S_0, I_1(0) = I_{10}, I_2(0) = I_{20} \ \ and \ \ A(0) = A_0$$

### 4.2.1 Model Assumptions

Here in the present study we develop a mathematical model to describe the population dynamics of HIV/AIDS disease based on the following assumptions:

(i) The population under study is heterogeneous and varying with time.

(ii) The whole human population is divided in to five classes.

(iii) The HIV can be transmitted by the sexual intercourse with infective peoples, vertical and blood borne transmissions.

(iv) The full blown AIDS class is sexually inactive.

(v) Assumed that the seropositive class could not transmit the disease.

(vi) All the new infected people are assumed to be initially unaware of the infection.

(vii) The probability of transferring the disease to susceptible population by unaware infected person is more than by aware infected person i.e $\beta_1 > \beta_2$.

(viii) The unaware infected people grow to AIDS much faster than the aware infected people i.e $\delta_1 > \delta_2$.

### 4.2.2 Compartments of population for the present model

In this section we have provided compartmentalization of the people. That is, the total population is divided into compartments. We have also described the flow of the people among these compartments. Notations and the description of the model parameters are also included. Flow diagram containing the compartments and flow directions is given for better understanding of the model. A dynamical system is constructed that describes the model. Mathematical analysis of the model is conducted. The mathematical modeling of the spread of HIV / AIDS disease among the population requires the whole human population to be divided in to five classes. The whole of the human population at any time t is a variable and is denoted by N(t). The five classes are as follows: (i) susceptible class the population size of this class at any time t is denoted by S(t) . The susceptible human has not yet infected by the disease but likely to get infected in future. (ii) Unaware infective class the population size of this class at any time t is denoted by $I_1(t)$. The unaware infective humans have already infected by the disease but they do not know that they were already infected. (iii) Aware infective class the population size of this class at any time t is denoted by $I_2(t)$. The aware infective humans have already infected by the disease and they know that they were already infected. (iv) AIDS class the population size of this class at any time t is denoted by A(t). The AIDS class people are already AIDS patients and (v) Seropositive class the population size of this class at any time t is denoted by $S_p(t)$. The Seropositive class people are HIV positive who are keeping themselves from unsafe sex and those who are taking ART treatment.

### 4.2.3 Flow of the People among the Compartments

People will join the susceptible compartment S(t) by natural birth. Some of these people will leave this compartment due to natural deaths and some others will go to $I_1(t)$ compartment after getting infected. The remaining people will stay in the S(t) compartment itself. People of S(t) compartment are likely to get infected by the people of $I_1(t)$ and $I_2(t)$ compartments only. But the people of AIDS compartment A(t) being physically too weak to participate in sexual activities, cannot transfer infection to susceptible people. In the present study the authors considered that the transfer of HIV from infected people to susceptible people is by sexual intercourse and transferring HIV by any other means like sharing needles; blood transfusion and the like. In to $I_1(t)$ compartment some people will enter from S(t) after getting infected,

some others will enter by immigration from other places and some more will enter by vertical transmission. From $I_1(t)$ compartment some people will go to $I_2(t)$ after becoming aware of the infection, some people will go to $S_p(t)$ because of keeping themselves well and unsafe sex, some will go to A(t) after conformation of full-fledged AIDS disease, some people will die with natural reasons, and others will stay back in $I_1(t)$ compartment itself. In to $I_2(t)$ compartment some people will enter from $I_1(t)$ after getting aware of the infection and some others will enter by immigration from other places. From $I_2(t)$ compartment some people will go to $S_p(t)$ after taking proper care for themselves and those who are taking ART, some people will go to A(t) after conformation of full-fledged AIDS disease, some people will die with natural reasons, and others will stay back in $I_2(t)$ compartment itself. In to $S_p(t)$ compartment people will enter from both $I_1(t)$ and $I_2(t)$ compartments after taking care of themselves and those who are taking ART. From $S_p(t)$ compartment people will leave when they die naturally. In to A(t) compartment people will enter from both $I_1(t)$ and $I_2(t)$ compartments after conformation of full-fledged AIDS disease. From A(t) compartment people will leave when they die naturally or die due to AIDS disease.

**Description of the Model Parameters**

We assume that the people are recruited into susceptible class at a constant rate of $Q_0$. This recruitment into the susceptible class is due to natural births. The people of susceptible class are likely to become infected through sexual contact and blood borne transmissions with the people of $I_1(t)$ and $I_2(t)$ classes. Thus, people from S(t) will go to $I_1(t)$ with a rate of $[\beta_1 I_1 + \beta_2 I_2]\frac{S}{N}$ and $\sigma[I_1+I_2]\frac{S}{N}$ respectively. Here the parameters $\beta_1$ and $\beta_2$ are the horizontal transmission rate to susceptible people by unaware and aware infective humans respectively and $\sigma$ is the rate of transmission of the disease by blood borne to susceptible people by unaware and aware infective humans respectively. Note that in this model we consider $\beta_1 > \beta_2$.That is, the probability of transferring the disease to susceptible population by unaware infected person is more than by aware infected person. People of S(t) after getting infected will initially go to $I_1(t)$ but not to $I_2(t)$. This is because, all the infected people are assumed to be initially unaware of the infection. Further, the people of S(t) compartment are assumed to die naturally with a rate of $\mu$. People will enter into$I_1(t)$ compartment from S(t) with a rate of $[\beta_1 I_1 + \beta_2 I_2]\frac{S}{N}$ and $\sigma[I_1 + I_2]\frac{S}{N}$, some others will enter due to immigration from other places at a rate of $p_1$ and some others will enter due to vertical transmission at a rate of $(1-\epsilon)\phi I_1$. It is assumed that the sexual contact between susceptible and unaware infected persons lead to the birth of infected children with a rate of $\phi$. Of these newly born but infected children a fraction $\epsilon$ dies during the birth due to infection and the remaining complementary fraction $(1-\epsilon)$ will enter into $I_1$ class.

From $I_1(t)$ compartment some people will go to $I_2(t)$ after becoming aware of the disease at a rate of $\theta$, and some others will go to A(t) compartment after confirmation of full AIDS disease at a rate of $\delta_1$ and some others will go to $S_p(t)$ compartment due to taking care of themselves at a rate of $k_1$. People of $I_1(t)$ compartment are assumed to die with natural reasons and leave the compartment at a rate of $\mu$. People will enter into $I_2(t)$ compartment from $I_1(t)$ after becoming aware of the disease with a rate of $\theta$ and some others will enter due to immigration from other places at a rate of $p_2$. People will go to $S_p(t)$ after taking care of themselves at a rate of $k_2$, people will go to A(t) compartment after confirmation of full AIDS disease at a rate of $\delta_2$. People of $I_2(t)$ compartment are assumed to die with natural reasons and leave the compartment at a rate of $\mu$. People will enter into $S_p(t)$ compartment from $I_1(t)$ and $I_2(t)$ compartments at a rate of $k_1$ and $k_2$ respectively. People of $S_p(t)$ compartment are assumed to die with natural reasons and leave the compartment at a rate of $\mu$. People will enter into A(t) compartment from $I_1(t)$ and $I_2(t)$ compartments at a rate of $\delta_1$ and $\delta_2$ respectively. Further, in this study we assume that $\delta_1 > \delta_2$ since the unaware infected people grow to AIDS much faster than the aware infected people. People of A(t) compartment are assumed to die with natural reasons at a rate of $\mu$ and die with AIDS disease at a rate of $\alpha$ and leave the compartment.

### 4.2.4 Flow Diagram of the Model

Using the above assumptions we developed the following flow diagram.

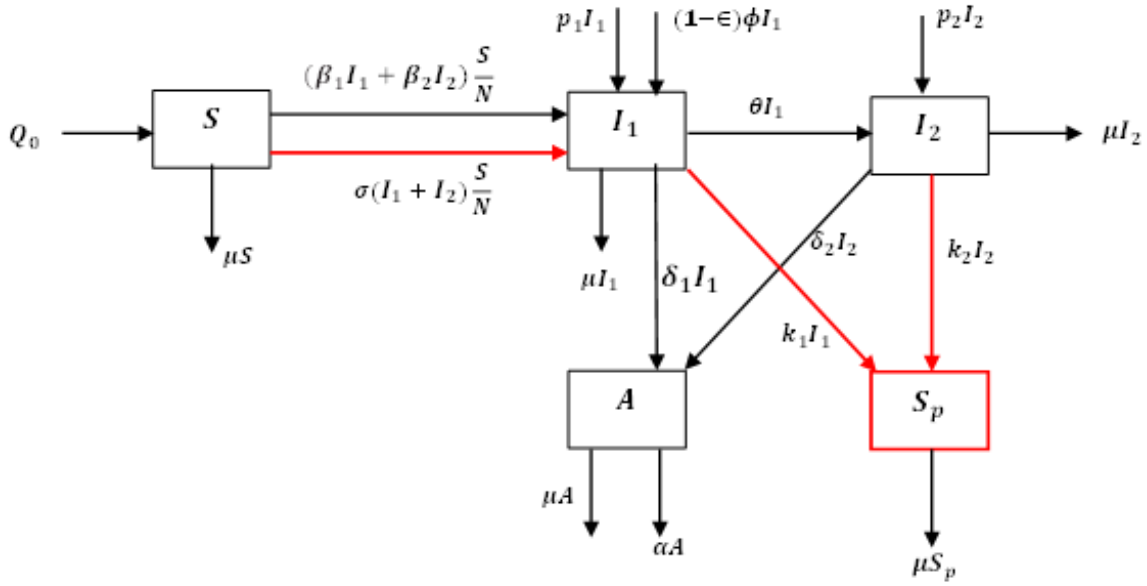


Figure 4.2: Flow diagram of the present model.

**Model Equations**

Based on the assumptions and the flow diagram of the present model, the dynamics of the HIV/AIDS transmission is governed by a system of non-linear ordinary differential equations as follows:

$$\frac{dS}{dt} = Q_0 - [\beta_1 I_1 + \beta_2 I_2]\frac{S}{N} - \sigma[I_1 + I_2]\frac{S}{N} - \mu S \tag{4.1}$$

$$\frac{dI_1}{dt} = [\beta_1 I_1 + \beta_2 I_2]\frac{S}{N} + \sigma[I_1 + I_2]\frac{S}{N} + p_1 I_1 + (1-\epsilon)\phi I_1 - (k_1 + \theta + \delta_1 + \mu)I_1 \tag{4.2}$$

$$\frac{dI_2}{dt} = p_2 I_2 + \theta I_1 - k_2 I_2 - (\delta_2 + \mu)I_2 \tag{4.3}$$

$$\frac{dS_p}{dt} = k_1 I_1 + k_2 I_2 - \mu S_p \tag{4.4}$$

$$\frac{dA}{dt} = \delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu)A \tag{4.5}$$

Here the initial conditions are considered to be

$$S(0) = S_0, I_1(0) = I_{10}, I_2(0) = I_{20}, S_p(0) = S_{p0} \;\; and \;\; A(0) = A_0 \tag{4.6}$$

### 4.2.5 Model properties

System (4.1−4.5) will be analyzed in a domain $\Omega \subset R_+^5$ where $\Omega = \{(S, I_1, I_2, S_p, A)\epsilon R_+^5\}$.

**Theorem 4.1.** *The solutions of system (4.1−4.5) with initial conditions satisfy (4.6)* $S(t) > 0, I_1(t) > 0, I_2(t) > 0, S_p(t) > 0, A(t) > 0$ *for all* $t > 0$. *The region* $\Omega \subset R_+^5$ *is positively invariant and attracting with respect to system (4.1−4.5).*

*Proof.* To show the positivity of the solution of the dynamical system comprising the equations (4.1−4.5), we have to consider and verify each differential equation and show that their solution is positive.

We define:

$\bar{t} = sup\{t > 0 : S(t) > 0, I_1(t) > 0, I_2(t) > 0, S_p(t) > 0 \; and \; A(t) > 0\}$

From the continuity of $S(t) > 0, I_1(t) > 0, I_2(t) > 0, S_p(t) > 0$ *and* $A(t) > 0$ , we deduce that $\bar{t} > 0$. Now if $\bar{t} = +\infty$, then the claim holds. That is, $S(t) > 0, I_1(t) > 0, I_2(t) > 0, S_p(t) > 0$ *and* $A(t) > 0$ for all $t > 0$. But if $0 < \bar{t} < +\infty$, from the definition of $\bar{t}$ it follows that,

$S(\bar{t}) = 0$ or $I_1(\bar{t}) = 0$ or $I_2(\bar{t}) = 0$ or $S_p(\bar{t}) = 0$ or $A(\bar{t})$=0

Now, first let us consider the differential equation (4.1) of the dynamical system

$\frac{dS}{dt} = Q_0 - [\beta_1 I_1 + \beta_2 I_2]\frac{S}{N} - \sigma[I_1 + I_2]\frac{S}{N} - \mu$

$\Rightarrow \frac{dS}{dt} + [q+\mu]S = Q_0$ where $q(t) = [(\beta_1+\sigma)I_1(t) + (\beta_2+\sigma)I_2(t)]\frac{1}{N(t)}$. This is a first order linear ordinary differential equation. Now we can find the integrating factor

$\mu_1(t) = e^{\int [q+\mu]dt} = e^{(Q(t)+\mu t)}$ where Q(t) is the anti-derivative of q(t). Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{(Q(t)+\mu t)}\frac{dS}{dt} + [q+\mu]e^{(Q(t)+\mu t)}S = Q_0 e^{(Q(t)+\mu t)}$

$\Rightarrow (e^{(Q(t)+\mu t)}S(t))' = Q_0 e^{(Q(t)+\mu t)}$

Integrating both sides

$\int_0^t (e^{(Q(s)+\mu s)}S(s))' ds = \int_0^t Q_0 e^{(Q(s)+\mu s)} ds$

$\Rightarrow e^{(Q(s)+\mu s)}S(s)|_0^t = \int_0^t Q_0 e^{(Q(s)+\mu s)} ds$

$\Rightarrow e^{(Q(t)+\mu t)}S(t) - e^{Q(0)}S(0) = \int_0^t Q_0 e^{(Q(s)+\mu s)} ds$

$\Rightarrow e^{(Q(t)+\mu t)}S(t) = e^{Q(0)}S(0) + \int_0^t Q_0 e^{(Q(s)+\mu s)} ds$

$\Rightarrow S(t) = \frac{e^{Q(0)}}{e^{(Q(t)+\mu t)}}S(0) + \frac{1}{e^{(Q(t)+\mu t)}}\int_0^t Q_0 e^{(Q(s)+\mu s)} ds$

$\Rightarrow S(t) = S(0)e^{(-Q(t)+Q(0)-\mu t)} + \int_0^t Q_0 e^{((Q(s)-Q(t))+\mu(s-t))} ds$

From this solution that S(t) is positive since $S(0) > 0, Q_0 > 0$ and the exponential function always positive.

Secondly let us consider the differential equation (4.2).

$\frac{dI_1}{dt} = [\beta_1 I_1 + \beta_2 I_2]\frac{S}{N} + \sigma[I_1 + I_2]\frac{S}{N} + p_1 I_1 + (1-\epsilon)\phi I_1 - (k_1 + \theta + \delta_1 + \mu)I_1$

$\Rightarrow \frac{dI_1}{dt} + [K - (\beta_1+\sigma)\frac{S}{N}]I_1 = (\beta_1+\sigma)\frac{I_2 S}{N}$ where $K = (k_1+\theta+\delta_1+\mu) - p_1 - (1-\epsilon)\phi$

This is a first order linear ordinary differential equation. We can find the integrating factor

$\Rightarrow \mu_1(t) = e^{\int [K-(\beta_1+\sigma)\frac{S}{N}]dt} = e^{(Kt-(\beta_1+\sigma)Q(t))}$ where Q(t) is the anti-derivative of $\frac{S(t)}{N(t)}$

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{(Kt-(\beta_1+\sigma)Q(t))}\frac{dI_1}{dt} + [K - \beta_1 + \sigma)\frac{S}{N}]e^{(Kt-(\beta_1+\sigma)Q(t))}I_1 = (\beta_1+\sigma)\frac{I_2 S}{N}e^{(Kt-(\beta_1+\sigma)Q(t)}$

$\Rightarrow (e^{(Kt-(\beta_1+\sigma)Q(t))}I_1(t))' = (\beta_1+\sigma)\frac{I_2 S}{N}e^{(Kt-(\beta_1+\sigma)Q(t))}$

Integrating both sides from 0 to $\bar{t}$ will give us

$\int_0^{\bar{t}} (e^{(Ks-(\beta_1+\sigma)Q(s))I_1(s)})' ds = \int_0^{\bar{t}} (\beta_1+\sigma)\frac{I_2 S}{N}e^{(Ks-(\beta_1+\sigma)Q(s))} ds$

$\Rightarrow e^{(Ks-(\beta_1+\sigma)Q(s))}I_1(s)|_0^{\bar{t}} = \int_0^{\bar{t}} (\beta_1+\sigma)\frac{I_2 S}{N}e^{(Ks-(\beta_1+\sigma)Q(s))} ds$

$\Rightarrow e^{(K\bar{t}-(\beta_1+\sigma)Q(\bar{t}))}I_1(t) - e^{-(\beta_1+\sigma)Q(0)}I_1(0) = \int_0^{\bar{t}} (\beta_1+\sigma)\frac{I_2 S}{N}e^{(Ks-(\beta_1+\sigma)Q(s))} ds$

$\Rightarrow e^{(K\bar{t}-(\beta_1+\sigma)Q(\bar{t}))}I_1(\bar{t}) = e^{-(\beta_1+\sigma)Q(0))}I_1(0) + \int_0^{\bar{t}} (\beta_1+\sigma)\frac{I_2 S}{N}e^{(Ks-(\beta_1+\sigma)Q(s))} ds$

$\Rightarrow I_1(\bar{t}) = \frac{e^{-(\beta_1+\sigma)Q(0)}}{e^{(K\bar{t}-(\beta_1+\sigma)Q(\bar{t}))}}I_1(0) + \frac{1}{e^{(Kt-(\beta_1+\sigma)Q(\bar{t}))}}\int_0^{\bar{t}} (\beta_1+\sigma)\frac{I_2 S}{N}e^{(Ks-(\beta_1+\sigma)Q(s))} ds$

$\Rightarrow I_1(\bar{t}) = I_1(0)e^{-K\bar{t}+(\beta_1+\sigma)Q(\bar{t})-(\beta_1+\sigma)Q(0))} + e^{-K\bar{t}+(\beta_1+\sigma)Q(\bar{t}))}\int_0^{\bar{t}} (\beta_1+\sigma)\frac{I_2 S}{N}e^{(Ks-(\beta_1+\sigma)Q(s))} ds$

since $I_1(0) > 0$ and from the definition of $\bar{t}$, we see that $S(t) > 0, I_2(t) > 0$ and also the exponential function always positive, then the solution $I_1(\bar{t}) > 0$. Hence, $I_1(\bar{t})$ could not be zero that is $I_1(\bar{t}) \neq 0$.

Thirdly, let us consider the differential equation (4.3)

$\frac{dI_2}{dt} = p_2 I_2 + \theta I_1 - k_2 I_2 - (\delta_2 + \mu) I_2$

$\Rightarrow \frac{dI_2}{dt} + hI_2 = \theta I_1$ where $h = (k_2 + \delta_2 + \mu - p_2)$

This is a first order linear ordinary differential equation. Now we can find the integrating factor

$\mu_1(t) = e^{\int h dt} = e^{ht}$

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{ht}\frac{dI_2}{dt} + he^{ht} I_2 = e^{ht}\theta I_1$

$\Rightarrow (e^{ht} I_2(t))' = e^{ht}\theta I_1$

Integrating both sides from 0 to $\bar{t}$ will give us

$\int_0^{\bar{t}} (e^{hs} I_2(s))' ds = \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow (e^{hs} I_2(s))|_0^{\bar{t}} = \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow e^{h\bar{t}} I_2(\bar{t}) - I_2(0) = \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow e^{h\bar{t}} I_2(\bar{t}) = I_2(0) + \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow I_2(\bar{t}) = \frac{1}{e^{h\bar{t}}} I_2(0) + \frac{1}{e^{h\bar{t}}} \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow I_2(\bar{t}) = I_2(0) e^{(-h\bar{t})} + e^{(-h\bar{t})} \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

since $I_2(0) > 0$ and from the definition of $\bar{t}$, we see that $I_1(t) > 0$ and also the exponential function always positive, then the solution $I_2(\bar{t}) > 0$. Hence, $I_2(\bar{t})$ could not be zero.

Fourthly, let us consider the differential equation (4.4).

$\frac{dS_p}{dt} = k_1 I_1 + k_2 I_2 - \mu S_p$

$\Rightarrow \frac{dS_p}{dt} + \mu I_2 = k_1 I_1 + k_2 I_2$

This is a first order linear ordinary differential equation.

Now we can find the integrating factor

$\mu_1(t) = e^{\int \mu dt} = e^{\mu t}$

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{\mu t}\frac{dS_p}{dt} + \mu e^{\mu t} S_p = e^{\mu t}(k_1 I_1 + k_2 I_2)$

$\Rightarrow (e^{\mu t} S_p)' = e^{\mu t}(k_1 I_1 + k_2 I_2)$

Integrating both sides from 0 to $\bar{t}$ will give us

$\int_0^{\bar{t}} (e^{\mu s} S_p(s))' ds = \int_0^{\bar{t}} e^{\mu s}(k_1 I_1 + k_2 I_2) ds$

$\Rightarrow (e^{\mu s} S_p(s))|_0^{\bar{t}} = \int_0^{\bar{t}} e^{\mu s}(k_1 I_1 + k_2 I_2) ds$

$\Rightarrow e^{\mu\bar{t}} S_p(\bar{t}) - S_p(0) = \int_0^{\bar{t}} e^{\mu s}(k_1 I_1 + k_2 I_2) ds$

$\Rightarrow e^{\mu\bar{t}} S_p(\bar{t}) = S_p(0) + \int_0^{\bar{t}} e^{\mu s}(k_1 I_1 + k_2 I_2) ds$

$\Rightarrow S_p(\bar{t}) = \frac{1}{e^{\mu\bar{t}}} S_p(0) + \frac{1}{e^{\mu\bar{t}}} \int_0^{\bar{t}} e^{\mu s}(k_1 I_1 + k_2 I_2) ds$

$\Rightarrow S_p(\bar{t}) = S_p(0)e^{-\mu\bar{t}} + e^{-\mu\bar{t}} \int_0^{\bar{t}} e^{\mu s}(k_1 I_1 + k_2 I_2)ds$

since $S_p(0) > 0$ and from the definition of $\bar{t}$, we see that $I_1(t) > 0, I_2(t) > 0$ and also the exponential function always positive, then the solution $S_p(\bar{t}) > 0$. Hence, $S_p(\bar{t})$ could not be zero.

Finally, let us consider the differential equation (4.5).

$\frac{dA}{dt} = \delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu)A$

$\Rightarrow \frac{dA}{dt} + (\alpha + \mu)A = \delta_1 I_1 + \delta_2 I_2$

This is a first order linear ordinary differential equation.

Now we can find the integrating factor

$\mu_1(t) = e^{\int(\alpha+\mu)dt} = e^{(\alpha+\mu)t}$

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^(\alpha + \mu)t\frac{dA}{dt} + (\alpha + \mu)e^{(\alpha+\mu)t}A = e^{(\alpha+\mu)t}(\delta_1 I_1 + \delta_2 I_2)$

$\Rightarrow (e^{(\alpha+\mu)t}A)' = e^{(\alpha+\mu)t}(\delta_1 I_1 + \delta_2 I_2)$

Integrating both sides from 0 to $\bar{t}$ will give us

$\int_0^{\bar{t}}(e^{(\alpha+\mu)s}A(s))'ds = \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2)ds$

$\Rightarrow (e^{(\alpha+\mu)s}A(s))|_0^{\bar{t}} = \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2)ds$

$\Rightarrow e^{(\alpha+\mu)\bar{t}}A(\bar{t}) - A(0) = \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2)ds$

$\Rightarrow e^{(\alpha+\mu)\bar{t}}A(\bar{t}) = A(0) + \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2)ds$

$\Rightarrow A(\bar{t}) = \frac{1}{e^{(\alpha+\mu)\bar{t}}}A(0) + \frac{1}{e^{(\alpha+\mu)\bar{t}}} \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2)ds$

$\Rightarrow A(\bar{t}) = A(0)e^{-(\alpha+\mu)\bar{t}} + e^{-(\alpha+\mu)\bar{t}} \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2)ds$

since $A(0) > 0$ and from the definition of $\bar{t}$, we see that $I_1(t) > 0, I_2(t) > 0$ and also the exponential function always positive, then the solution $A(\bar{t}) > 0$. Hence, $A(\bar{t})$ could not be zero.

Therefore all the state variables at $\bar{t}$ could not be zero, implies that $\bar{t}$ is not finite. Consequently $\bar{t} = +\infty$, so that for all $t \geq 0$, $S(t) > 0, I_1(t) > 0, I_2(t) > 0, S_p(t) > 0$, and $A(t) > 0$. By this we have shown that all the solutions of system (4.1) to (4.5) are in $R_+^5$, provided that the initial conditions are positive. □

We now show that all feasible solutions are uniformly bounded in $\Omega$.

**Theorem 4.2.** *The feasible region $\Omega$ of the system (4.1) to (4.5) is defined as:*

$\Omega = \{(S(t), I_1(t), I_2(t), S_p(t), A(t))\epsilon R_+^5 : 0 < N(t) \leq \frac{Q_0}{\mu}\}$

*Proof.* We assume that all state variables and parameters are positive.

Here we have $N = S + I_1 + I_2 + S_p + A$ then

$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI_1}{dt} + \frac{dI_2}{dt} + \frac{dS_p}{dt} + \frac{dA}{dt}$

Summing up all the five equations from systems (4.1) to (4.5) and assuming the inequality $p_1 I_1 + (1-\epsilon)\phi I_1 + p_2 I_2 \leq \alpha A$ we obtain $\frac{dN}{dt} \leq Q_0 - \mu N$

$\Rightarrow \frac{dN}{Q_0 - \mu N} \leq dt$ integrating both sides

$\int_0^t \frac{dN}{Q_0 - \mu N} \leq \int_0^t ds \Rightarrow \frac{-1}{\mu}[\ln(Q_0 - \mu N(t)) - \ln(Q_0 - \mu N(0))] \leq t$

$\Rightarrow [\ln(Q_0 - \mu N(t)) - \ln(Q_0 - \mu N(0))] \geq -\mu t$

$\Rightarrow \ln[\frac{Q_0 - \mu N(t)}{Q_0 - \mu N(0)}] \geq -\mu t$

$\Rightarrow [\frac{Q_0 - \mu N(t)}{Q_0 - \mu N(0)}] \geq e^{-\mu t}$

$\Rightarrow Q_0 - \mu N(t) \geq e^{-\mu t}(Q_0 - \mu N(0))$

$\Rightarrow Q_0 - \mu N(t) \geq Q_0 e^{-\mu t} - \mu N(0) e^{-\mu t}$

$\Rightarrow \mu N(t) \leq Q_0 - Q_0 e^{-\mu t} + \mu N(0) e^{-\mu t} \leq Q_0 + \mu N(0) e^{-\mu t}$

$\Rightarrow N(t) \leq \frac{Q_0}{\mu} + N(0) e^{-\mu t}$

Thus as $t \longrightarrow \infty$ we have $0 < N(t) \leq \frac{Q_0}{\mu}$ which indicates that the total population is bounded.

□

## 4.3 Stability Analysis of Disease Free and Endemic Equilibrium Points

In this section we identify the equilibrium points of the model developed in this study and provided as a system of equations from (4.1) to (4.5). We also analyze their stability conditions and present the results. The system exhibits two types of equilibrium points; disease free equilibrium point and endemic equilibrium point.

### 4.3.1 Disease Free Equilibrium Point

The disease free equilibrium of the model (4.1) to (4.5), is obtained by setting $\frac{dS}{dt} = \frac{dI_1}{dt} = \frac{dI_2}{dt} = \frac{dS_p}{dt} = \frac{dA}{dt} = 0$. Further at the disease free equilibrium point there are neither infective people

nor AIDS patients. Then

$$\frac{dS}{dt} = Q_0 - [\beta_1 I_1 + \beta_2 I_2]\frac{S}{N} - \sigma[I_1 + I_2]\frac{S}{N} - \mu S = 0$$
$$\frac{dI_1}{dt} = [\beta_1 I_1 + \beta_2 I_2]\frac{S}{N} + \sigma[I_1 + I_2]\frac{S}{N} + p_1 I_1 + (1-\epsilon)\phi I_1 - (k_1 + \theta + \delta_1 + \mu)I_1 = 0$$
$$\frac{dI_2}{dt} = p_2 I_2 + \theta I_1 - k_2 I_2 - (\delta_2 + \mu)I_2 = 0$$
$$\frac{dS_p}{dt} = k_1 I_1 + k_2 I_2 - \mu S_p = 0$$
$$\frac{dA}{dt} = \delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu)A = 0$$

This system reduced to

$Q_0 - \mu S = 0$ Since at disease free we have $I_1 = I_2 = A = S_p = 0$. Then $Q_0 - \mu S = 0$

$\Rightarrow \mu S = Q_0$

$S = \frac{Q_0}{\mu}$

Thus the disease free equilibrium of the model is given by $E_0 = (\frac{Q_0}{\mu}, 0, 0, 0, 0)$.

**Basic Reproduction number**

The reproduction number is defined as the average number of secondary cases produced by a typical infected individual during his or her entire life as infectious or infectious period when introduced or allowed to live in a population of susceptible [26]. We shall now compute the basic reproduction number of the present model using the next generation method. The basic reproduction number is a threshold quantity used to study the spread of an infection disease in epidemiological modeling and it is the spectral radius of the next generation matrix. It is defined as $R_0 = \rho(FV^{-1})$ here $FV^{-1} = [\frac{\partial F_i(x_0))}{\partial x_j}][\frac{\partial V_i(x_0)}{\partial x_j}]^{-1}$ where $F_i$ is the rate of appearance of new infections in the compartment i; $V_i$ is the transfer of individuals in and out of compartment i.

Here we consider the disease manifests compartment for simplification of our work.

$$F_i(S, I_1, I_2, S_p, A) = \begin{bmatrix} f_1(S, I_1, I_2, S_p, A) \\ g_1(S, I_1, I_2, S_p, A) \end{bmatrix}$$
$$= \begin{bmatrix} [\beta_1 I_1 + \beta_2 I_2]\frac{S}{N} + \sigma[I_1 + I_2]\frac{S}{N} \\ 0 \end{bmatrix}$$
$$V_i(S, I_1, I_2, S_p, A) = \begin{bmatrix} f_2(S, I_1, I_2, S_p, A) \\ g_2(S, I_1, I_2, S_p, A) \end{bmatrix}$$

$$= \begin{bmatrix} (k_1+\theta+\delta_1+\mu)I_1 - (p_1+(1-\epsilon)\phi)I_1 \\ -\theta I_1 - p_2 I_2 + (k_2+\delta_2+\mu)I_2 \end{bmatrix}$$

$$F = DF_i = \begin{bmatrix} \frac{\partial f_1}{\partial I_1} & \frac{\partial f_1}{\partial I_2} \\ \frac{\partial g_1}{\partial I_1} & \frac{\partial g_1}{\partial I_2} \end{bmatrix}$$

$$= \begin{bmatrix} [(\beta_1+\sigma)\frac{S}{N} & (\beta_2+\sigma)\frac{S}{N} \\ 0 & 0 \end{bmatrix}$$

At disease free equilibrium point we have $S \approx N$. Thus

$$F = \begin{bmatrix} (\beta_1+\sigma) & (\beta_2+\sigma) \\ 0 & 0 \end{bmatrix}$$

$$V = DV_i = \begin{bmatrix} \frac{\partial f_2}{\partial I_1} & \frac{\partial f_2}{\partial I_2} \\ \frac{\partial g_2}{\partial I_1} & \frac{\partial g_2}{\partial I_2} \end{bmatrix}$$

$$= \begin{bmatrix} (k_1+\theta+\delta_1+\mu) - (p_1+(1-\epsilon)\phi) & 0 \\ -\theta & (k_2+\delta_2+\mu) - p_2 \end{bmatrix}$$

To get $V^{-1}$, we use the adjoint matrix method.

$V^{-1} = \frac{1}{det(V)} adj(V)$

Where

$$det(V) = \begin{vmatrix} (k_1+\theta+\delta_1+\mu) - (p_1+(1-\epsilon)\phi) & 0 \\ -\theta & (k_2+\delta_2+\mu) - p_2 \end{vmatrix}$$

$$= (k_1+\theta+\delta_1+\mu-(p_1+(1-\epsilon)\phi))(k_2+\delta_2+\mu-p_2)$$

$$adj(V) = \begin{bmatrix} k_2+\delta_2+\mu-p_2 & 0 \\ \theta & k_1+\theta+\delta_1+\mu-(p_1+(1-\epsilon)\phi) \end{bmatrix}$$

Let $\Delta_1 = k_1+\theta+\delta_1+\mu-p_1-(1-\epsilon)\phi$ *and* $\Delta_2 = k_2+\delta_2+\mu-p_2$

Then

$$V^{-1} = \frac{1}{\Delta_1\Delta_2} \begin{bmatrix} \Delta_2 & 0 \\ \theta & \Delta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\Delta_1} & 0 \\ \frac{\theta}{\Delta_1\Delta_2} & \frac{1}{\Delta_2} \end{bmatrix}$$

thus

$$FV^{-1} = \begin{bmatrix} \beta_1+\sigma & \beta_2+\sigma \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\Delta_1} & 0 \\ \frac{\theta}{\Delta_1\Delta_2} & \frac{1}{\Delta_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\beta_1+\sigma}{\Delta_1} + \frac{\theta(\beta_2+\sigma)}{\Delta_1\Delta_2} & \frac{\beta_2+\sigma}{\Delta_2} \\ 0 & 0 \end{bmatrix}$$

We find the eigenvalues of $FV^{-1}$ by solving the characteristic equation $|FV^{-1} - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} \frac{\beta_1+\sigma}{\Delta_1} + \frac{\theta(\beta_2+\sigma)}{\Delta_1\Delta_2} - \lambda & \frac{\beta_2+\sigma}{\Delta_2} \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)(\frac{\beta_1+\sigma}{\Delta_1} + \frac{\theta(\beta_2+\sigma)}{\Delta_1\Delta_2} - \lambda) = 0$$

$\Rightarrow \lambda_1 = 0$ and $\lambda_2 = \frac{\beta_1+\sigma}{\Delta_1} + \frac{\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}$ thus the spectral radius of $FV^{-1}$ is given by $R_0 = \max[\lambda_1, \lambda_2] = \lambda_2$

Therefore the basic reproduction number of the model is

$$R_0 = \frac{\beta_1+\sigma}{\Delta_1} + \frac{\theta(\beta_2+\sigma)}{\Delta_1\Delta_2} = \frac{(k_2+\delta_2+\mu-p_2)(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{(k_1+\theta+\delta_1+\mu-p_1-(1-\epsilon)\phi)(k_2+\delta_2+\mu-p_2)}$$

In order to assess the contribution of unaware and aware infected population on the dynamics of HIV/AIDS, let us divide the basic reproduction number $R_0$ of the present model in to the reproduction numbers of both unaware $R_{0U}$ and aware $R_{0A}$ infected population independently. That is $R_0 = R_{0U} + R_{0A}$ where $R_{0U} = \frac{\beta_1+\sigma}{\Delta_1}$, $R_{0A} = \frac{\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}$. This can also be more analyzed as follows.

If we assume that the rates of transmissions of the disease from unaware infective are equal to zero (i.e $\beta_1 = 0$ and $\sigma = 0$), then the reproduction number $R_0$ becomes

$$R_0 = \frac{(k_2+\delta_2+\mu-p_2)(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{(k_1+\theta+\delta_1+\mu-p_1-(1-\epsilon)\phi)(k_2+\delta_2+\mu-p_2}$$

$$= \frac{\theta(\beta_2+\sigma)}{(k_1+\theta+\delta_1+\mu-p_1-(1-\epsilon)\phi)(k_2+\delta_2+\mu-p_2)}$$

$$\Rightarrow R_0 = R_{0A} = \frac{\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}$$

This is the reproduction number of aware infected population.

Similarly, if we assume that the rate of transmissions of the disease from aware infective are equal to zero (i.e $\beta_2 = 0$ and $\sigma = 0$), then the reproduction number $R_0$ becomes

$$R_0 = \frac{(k_2 + \delta_2 + \mu - p_2)(\beta_1 + \sigma) + \theta(\beta_2 + \sigma)}{(k_1 + \theta + \delta_1 + \mu - p_1 - (1-\epsilon)\phi)(k_2 + \delta_2 + \mu - p_2}$$

$$= \frac{(\beta_1 + \sigma)}{(k_1 + \theta + \delta_1 + \mu - p_1 - (1-\epsilon)\phi)}$$

$$\Rightarrow R_0 = R_{0U} = \frac{\beta_1 + \sigma}{\Delta_1}$$

This is the reproduction number of unaware infected population.

We now investigate the local and global stability of the disease free equilibrium point.

**Local stability of the disease free equilibrium point $E_0$**

**Theorem 4.3.** *The disease free equilibrium point $E_0$ of the system of ordinary differential equations (4.1) to (4.5) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

*Proof.* Initially at $t = 0$, $S(0) > 0, I_1(0) \geq 0, I_2(0) \geq 0, S_p(0) \geq 0$ this means initially there is no AIDS patient. Hence, we only consider the subsystem of four equations (4.1), (4.2), (4.3) and (4.4). The Jacobian matrix associated with the subsystem equations at the disease free equilibrium point $E_0 = (\frac{Q_0}{\mu}, 0, 0, 0, 0)$ is given by:

$$J(E_0) = \begin{bmatrix} -\mu & -(\beta_1 + \sigma) & -(\beta_2 + \sigma) & 0 \\ 0 & (\beta_1 + \sigma) - \Delta_1 & (\beta_2 + \sigma) & 0 \\ 0 & \theta & -\Delta_2 & 0 \\ 0 & k_1 & k_2 & -\mu \end{bmatrix}$$

The characteristic equation $|J(E_0) - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\mu - \lambda & -(\beta_1 + \sigma) & -(\beta_2 + \sigma) & 0 \\ 0 & (\beta_1 + \sigma) - \Delta_1 - \lambda & (\beta_2 + \sigma) & 0 \\ 0 & \theta & -\Delta_2 - \lambda & 0 \\ 0 & k_1 & k_2 & -\mu - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\mu + \lambda) \begin{vmatrix} (\beta_1 + \sigma) - \Delta_1 - \lambda & (\beta_2 + \sigma) & 0 \\ \theta & -\Delta_2 - \lambda & 0 \\ k_1 & k_2 & -\mu - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\mu+\lambda)^2 \begin{vmatrix} (\beta_1+\sigma)-\Delta_1-\lambda & (\beta_2+\sigma) \\ \theta & -\Delta_2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\mu+\lambda)^2[(\beta_1+\sigma-\Delta_1-\lambda)(-\Delta_2-\lambda)-\theta(\beta_2+\sigma)] = 0$$

$$\Rightarrow (\mu+\lambda)^2[\lambda^2-\lambda(\beta_1+\sigma-\Delta_1)+\lambda\Delta_2-(\beta_1+\sigma-\Delta_1)\Delta_2-\theta(\beta_2+\sigma)] = 0$$

$$\Rightarrow (\mu+\lambda)^2[\lambda^2+(-\beta_1-\sigma+\Delta_1+\Delta_2)\lambda+\Delta_1\Delta_2-(\beta_1+\sigma)\Delta_2-\theta(\beta_2+\sigma)] = 0$$

$$\Rightarrow (\mu+\lambda)^2[\lambda^2+B\lambda+C] = 0$$

Where $C = \Delta_1\Delta_2-(\beta_1+\sigma)\Delta_2-\theta(\beta_2+\sigma)$ and $B = (-\beta_1-\sigma+\Delta_1+\Delta_2)$
If $R_0 < 1$ implies $\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2} < 1 \Rightarrow \Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma) < \Delta_1\Delta_2$

$$\Rightarrow \Delta_1\Delta_2-\Delta_2(\beta_1+\sigma)-\theta(\beta_2+\sigma) > 0$$

$$\Rightarrow C > 0$$

Since $\Delta_1\Delta_2-\Delta_2(\beta_1+\sigma)-\theta(\beta_2+\sigma) > 0$ we have $(\Delta_1-(\beta_1+\sigma))\Delta_2-\theta(\beta_2+\sigma) > 0$

$$\Rightarrow \Delta_1-(\beta_1+\sigma) > 0$$

$$\Rightarrow \Delta_1-(\beta_1+\sigma)+\Delta_2 > 0$$

$$\Rightarrow B > 0$$

Therefore the quadratic equation $\lambda^2+B\lambda+C = 0$ has two negative real roots.
In general we have all eigenvalues of the jacobian matrix negative. Hence the disease free equilibrium point is locally asymptotically stable.
If $R_0 > 1$ then the characteristic equation will have positive eigenvalues, so $E_0$ is unstable. □

**Global stability of the disease free equilibrium point** $E_0$

**Theorem 4.4.** *the disease free equilibrium point $E_0$ is globally asymptotically stable if $R_0 < 1$.*

*Proof.* we now construct a Lyapunov function

$$V = \alpha_1 I_1+\alpha_2 I_2+\alpha_3 S_p+\alpha_4 A$$

Where $\alpha_i, i = 1,2,3,4$ are positive constants to be determined. The time derivative of V is given by

$$\frac{dV}{dt} = \alpha_1\frac{dI_1}{dt}+\alpha_2\frac{dI_2}{dt}+\alpha_3\frac{dS_p}{dt}+\alpha_4\frac{dA}{dt}$$

$$
\begin{aligned}
=&\alpha_1[(\frac{\beta_1+\sigma}{N})SI_1+(\frac{\beta_2+\sigma}{N})SI_2+(p_1+(1-\epsilon)\phi-(k_1+\theta+\delta_1+\mu))I_1]\\
&+\alpha_2[(p_2-(k_2+\delta_2+\delta))I_2+\theta I_1]+\alpha_3[k_1I_1+k_2I_2-\mu S_p]+\alpha_4[\delta_1I_1+\delta_2I_2-(\alpha+\mu)A]\\
=&\alpha_1[(\frac{\beta_1+\sigma}{N})S+(p_1+(1-\epsilon)\phi-(k_1+\theta+\delta_1+\mu))]I_1+\alpha_1(\frac{\beta_2+\sigma}{N})SI_2+\alpha_2(p_2-(k_2+\delta_2+\delta))I_2\\
&+\alpha_2\theta I_1+\alpha_3k_1I_1+\alpha_3k_2I_2-\alpha_3\mu S_p+\alpha_4\delta_1I_1+\alpha_4\delta_2I_2-\alpha_4(\alpha+\mu)A\\
=&\alpha_1[(\frac{\beta_1+\sigma}{N})S-\Delta_1)]I_1+\alpha_2\theta I_1+\alpha_3k_1I_1+\alpha_4\delta_1I_1+\alpha_1(\frac{\beta_2+\sigma}{N})SI_2-\alpha_2\Delta_2I_2+\alpha_3k_2I_2+\alpha_4\delta_2I_2\\
&-\alpha_3\mu S_p-\alpha_4(\alpha+\mu)A\\
=&[\alpha_1((\frac{\beta_1+\sigma}{N})S-\Delta_1)+\alpha_2\theta+\alpha_3k_1+\alpha_4\delta_1]I_1+\alpha_1[(\frac{\beta_2+\sigma}{N})S-\alpha_2\Delta_2+\alpha_3k_2+\alpha_4\delta_2]I_2\\
&-\alpha_3\mu S_p-\alpha_4(\alpha+\mu)A\\
\leq&[\alpha_1((\beta_1+\sigma)-\Delta_1)+\alpha_2\theta+\alpha_3k_1+\alpha_4\delta_1]I_1+\alpha_1[(\beta_2+\sigma)-\alpha_2\Delta_2+\alpha_3k_2+\alpha_4\delta_2]I_2\\
&-\alpha_3\mu S_p-\alpha_4(\alpha+\mu)A
\end{aligned}
$$

Take the coefficients of $I_2, S_p$ and A are equal to zero. Then we get

$$
\begin{aligned}
&-\alpha_3\mu=0\Rightarrow\alpha_3=0\\
&-\alpha_4(\alpha+\mu)=0\Rightarrow\alpha_4=0\\
&\alpha_1(\beta_2+\sigma)-\alpha_2\Delta_2+\alpha_3k_2+\alpha_4\delta_2=0\\
&\Rightarrow\alpha_1(\beta_2+\sigma)-\alpha_2\\
&\Delta_2=0\ \textit{since}\ \alpha_3=\alpha_4=0\\
&\Rightarrow\alpha_1(\beta_2+\sigma)=\alpha_2\Delta_2\\
&\Rightarrow\alpha_2=\frac{\alpha_1(\beta_2+\sigma)}{\Delta_2}
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{dV}{dt}\leq[\alpha_1((\beta_1+\sigma)-\Delta_1)+\alpha_2\theta]I_1\\
&\Rightarrow\frac{dV}{dt}\leq[\alpha_1((\beta_1+\sigma)-\Delta_1)+\frac{\alpha_1\theta(\beta_2+\sigma)}{\Delta_2}]I_1\\
&\Rightarrow\frac{dV}{dt}\leq[\alpha_1(\beta_1+\sigma)-\alpha_1\Delta_1+\frac{\alpha_1\theta(\beta_2+\sigma)}{\Delta_2}]I_1\\
&\Rightarrow\frac{dV}{dt}\leq[\alpha_1\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_2}-\alpha_1\Delta_1]I_1\\
&\Rightarrow\frac{dV}{dt}\leq[\alpha_1\frac{R_0\Delta_1\Delta_2}{\Delta_2}-\alpha_1\Delta_1]I_1\\
&\Rightarrow\frac{dV}{dt}\leq[\alpha_1R_0\Delta_1-\alpha_1\Delta_1]I_1\\
&\Rightarrow\frac{dV}{dt}\leq\alpha_1\Delta_1[R_0-1]I_1
\end{aligned}
$$

We note that $\frac{dV}{dt} \leq 0$ if $R_0 < 1$.

Furthermore, $\frac{dV}{dt} = 0$ if and only if $I_1 = I_2 = S_p = A = 0$. Therefore, the largest compact invariant set in $(S, I_1, I_2, S_p, A)\epsilon\Omega : \frac{dV}{dt} = 0$, where $R_0 < 1$ is the singleton $\{E_0\}$. LaSalle's (1976) invariance principle then implies that $E_0$ is globally stable in $\Omega$ if $R_0 < 1$ otherwise it is unstable. □

### 4.3.2 Endemic Equilibrium Point

We consider the system equations (4.1) to (4.5). At the endemic equilibrium point $E^* = (S^*, I_1^*, I_2^*, S_p^*, A^*)$, we set each right hand side in system equations to zero and express each dependent variable in terms of $I_1^*$ at the equilibrium point.

$$\begin{aligned}
\frac{dS}{dt} &= Q_0 - [\beta_1 I_1 + \beta_2 I_2]\frac{S}{N} - \sigma[I_1 + I_2]\frac{S}{N} - \mu S = 0 \\
\frac{dI_1}{dt} &= [\beta_1 I_1 + \beta_2 I_2]\frac{S}{N} + \sigma[I_1 + I_2]\frac{S}{N} + p_1 I_1 + (1-\epsilon)\phi I_1 - (k_1 + \theta + \delta_1 + \mu)I_1 = 0 \\
\frac{dI_2}{dt} &= p_2 I_2 + \theta I_1 - k_2 I_2 - (\delta_2 + \mu)I_2 = 0 \\
\frac{dS_p}{dt} &= k_1 I_1 + k_2 I_2 - \mu S_p = 0 \\
\frac{dA}{dt} &= \delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu)A = 0
\end{aligned}$$

Solving equation (4.1) and (4.2) we get

$$Q_0 - \mu S^* + p_1 I_1^* + (1-\epsilon)\phi I_1^* - (k_1 + \theta + \delta_1 + \mu)I_1^* = 0$$

$$\Rightarrow \mu S^* = Q_0 + p_1 I_1^* + (1-\epsilon)\phi I_1^* - (k_1 + \theta + \delta_1 + \mu)I_1^*$$

$$\Rightarrow S^* = \frac{Q_0 + [p_1 + (1-\epsilon)\phi - (k_1 + \theta + \delta_1 + \mu)]I_1^*}{\mu}$$

$$\Rightarrow S^* = \frac{Q_0 - \Delta_1 I_1^*}{\mu}$$

From the equation (4.3) we have

$$p_2 I_2^* + \theta I_1^* - k_2 I_2^* - (\delta_2 + \mu)I_2^* = 0$$

$$\Rightarrow (k_2 + \delta_2 + \mu - p_2)I_2^* = \theta I_1^*$$

$$\Rightarrow I_2^* = \frac{\theta I_1^*}{(k_2 + \delta_2 + \mu - p_2)}$$

$$\Rightarrow I_2^* = \frac{\theta I_1^*}{\Delta_2}$$

From equation (4.1)we have

$$Q_0 - [\beta_1 I_1^* + \beta_2 I_2^*]\frac{S^*}{N^*} - \sigma[I_1^* + I_2^*]\frac{S^*}{N^*} - \mu S^* = 0$$

$$\Rightarrow Q_0 = [\beta_1 I_1^* + \beta_2 I_2^*]\frac{S^*}{N^*} + \sigma[I_1^* + I_2^*]\frac{S^*}{N^*} + \mu S^*$$

$$\Rightarrow ([\beta_1 I_1^* + \beta_2 I_2^*]\frac{1}{N^*} + \sigma[I_1^* + I_2^*]\frac{1}{N^*} + \mu)S^* = Q_0$$

$$\Rightarrow ([\beta_1 I_1^* + \beta_2 \frac{\theta I_1^*}{\Delta_2}]\frac{1}{N^*} + \sigma[I_1^* + \frac{\theta I_1^*}{\Delta_2}]\frac{1}{N^*} + \mu)(\frac{Q_0 - \Delta_1 I_1^*}{\mu}) = Q_0$$

$$\Rightarrow (\frac{\beta_1 I_1^* \Delta_2 + \beta_2 \theta I_1^*}{\Delta_2} + \frac{\sigma \Delta_2 I_1^* + \sigma\theta I_1^*}{\Delta_2} + \mu N^*)(\frac{Q_0 - \Delta_1 I_1^*}{\mu}) = Q_0 N^*$$

$$\Rightarrow (\frac{\beta_1 \Delta_2 + \beta_2 \theta}{\Delta_2})\frac{Q_0}{\mu} I_1^* + (\frac{\sigma\Delta_2 + \sigma\theta}{\Delta_2})\frac{Q_0}{\mu} I_1^* + Q_0 N^* - \Delta_1 (\frac{\beta_1\Delta_2 + \beta_2\theta + \sigma\Delta_2 + \sigma\theta}{\mu\Delta_2})(I_1^*)^2$$

$$- \Delta_1 N^* I_1^* = Q_0 N^*$$

$$\Rightarrow (\frac{\beta_1 \Delta_2 + \beta_2 \theta}{\Delta_2})\frac{Q_0}{\mu} I_1^* + (\frac{\sigma\Delta_2 + \sigma\theta}{\Delta_2})\frac{Q_0}{\mu} I_1^* - \Delta_1 (\frac{\beta_1\Delta_2 + \beta_2\theta + \sigma\Delta_2 + \sigma\theta}{\mu\Delta_2})(I_1^*)^2 - \Delta_1 N^* I_1^* = 0$$

$$\Rightarrow I_1^* [(\frac{\beta_1 \Delta_2 + \beta_2 \theta}{\Delta_2})\frac{Q_0}{\mu} + (\frac{\sigma\Delta_2 + \sigma\theta}{\Delta_2})\frac{Q_0}{\mu} - \Delta_1 (\frac{\beta_1\Delta_2 + \beta_2\theta + \sigma\Delta_2 + \sigma\theta}{\mu\Delta_2}) I_1^* - \Delta_1 N^*] = 0$$

$\Rightarrow I_1^* = 0$ *which is the DFE or*

$$I_1^* = [-\Delta_1 N^* + (\frac{\beta_1\Delta_2 + \beta_2\theta + \sigma\Delta_2 + \sigma\theta}{\Delta_2})\frac{Q_0}{\mu}][\frac{\mu\Delta_2}{\Delta_1(\beta_1\Delta_2 + \beta_2\theta + \sigma\Delta_2 + \sigma\theta)}]$$

$$\Rightarrow I_1^* = [\frac{(\beta_1\Delta_2 + \beta_2\theta + \sigma\Delta_2 + \sigma\theta)Q_0 - \mu\Delta_1\Delta_2 N^*}{\mu\Delta_2}][\frac{\mu\Delta_2}{\Delta_1(\beta_1\Delta_2 + \beta_2\theta + \sigma\Delta_2 + \sigma\theta)}]$$

$$\Rightarrow I_1^* = \frac{\mu}{\Delta_1}[\frac{Q_0}{\mu} - \frac{\Delta_1\Delta_2 N^*}{\beta_1\Delta_2 + \beta_2\theta + \sigma\Delta_2 + \sigma\theta}]$$

$$\Rightarrow I_1^* = \frac{\mu}{\Delta_1}[\frac{Q_0}{\mu} - \frac{N^*}{R_0}]$$

$$\Rightarrow I_1^* = \frac{\mu}{\Delta_1}[\frac{Q_0}{\mu} - \frac{\Delta_1\Delta_2 N^*}{\beta_1\Delta_2 + \beta_2\theta + \sigma\Delta_2 + \sigma\theta}]$$

$$\Rightarrow I_1^* = \frac{Q_0}{\Delta_1}[1 - \frac{1}{R_0}]$$

From this we can observe that $I_1^*$ will be positive if $R_0 > 1$. Thus the unique endemic equilibrium exists whenever $R_0 > 1, \Delta_1 > 0$ and $\Delta_2 > 0$. The endemic equilibrium is:

$S^* = \frac{Q_0 - \Delta_1 \frac{Q_0}{\Delta_1}\left(1 - \frac{1}{R_0}\right)}{\mu} = \frac{N^*}{R_0}$

$I_1^* = \frac{Q_0}{\Delta_1}(1 - \frac{1}{R_0})$

$I_2{}^* = \frac{\theta\frac{Q_0}{\Delta_1}\left(1 - \frac{1}{R_0}\right)}{\Delta_2} = \frac{Q_0\theta\left(1 - \frac{1}{R_0}\right)}{\Delta_1\Delta_2}$

From the fourth equation of the system we have

$k_1 I_1^* + k_2 I_2^* - \mu S_p^* = 0$

$\Rightarrow \mu S_p^* = k_1 I_1^* + k_2 I_2^*$

$\Rightarrow S_p^* = \frac{k_1 I_1^* + k_2 I_2^*}{\mu} = \frac{k_1\frac{Q_0}{\Delta_1}(1 - \frac{1}{R_0}) + k_2\frac{Q_0\theta(1 - \frac{1}{R_0})}{\Delta_1\Delta_2}}{\mu}$

$\Rightarrow S_p^* = \frac{k_1\Delta_2 Q_0\left(1-\frac{1}{R_0}\right)+k_2 Q_0\theta\left(1-\frac{1}{R_0}\right)}{\mu\Delta_1\Delta_2}$

From the fifth equation of the system we have

$\delta_1 I_1^* + \delta_2 I_2^* - (\alpha+\mu)A^* = 0$

$\Rightarrow (\alpha+\mu)A^* = \delta_1 I_1^* + \delta_2 I_2^*$

$\Rightarrow A^* = \frac{\delta_1 I_1^* + \delta_2 I_2^*}{(\alpha+\mu)} = \frac{\delta_1 \frac{Q_0}{\Delta_1}\left(1-\frac{1}{R_0}\right)+\delta_2\frac{Q_0\theta\left(1-\frac{1}{R_0}\right)}{\Delta_1\Delta_2}}{(\alpha+\mu)}$

$\Rightarrow A^* = \frac{\delta_1\Delta_2 Q_0\left(1-\frac{1}{R_0}\right)+\delta_2 Q_0\theta\left(1-\frac{1}{R_0}\right)}{(\alpha+\mu)\Delta_1\Delta_2}$

In general the unique endemic equilibrium point $E^* = (S^*, I_1^*, I_2^*, S_p^*, A^*)$ of the system (4.1) to (4.5) is:

$$S^* = \frac{N}{R_0}$$

$$I_1{}^* = \frac{Q_0}{\Delta_1}\left(1-\frac{1}{R_0}\right)$$

$$I_2{}^* = \frac{Q_0\theta\left(1-\frac{1}{R_0}\right)}{\Delta_1\Delta_2}$$

$$S_p^* = \frac{k_1\Delta_2 Q_0\left(1-\frac{1}{R_0}\right)+k_2 Q_0\theta\left(1-\frac{1}{R_0}\right)}{\mu\Delta_1\Delta_2}$$

$$A^* = \frac{\delta_1\Delta_2 Q_0\left(1-\frac{1}{R_0}\right)+\delta_2 Q_0\theta\left(1-\frac{1}{R_0}\right)}{(\alpha+\mu)\Delta_1\Delta_2}$$

**Local stability of Endemic equilibrium point**

We now investigate the local stability of the endemic equilibrium point $E^*$.

**Theorem 4.5.** *The positive endemic equilibrium point $E^*$ of the system of equations (4.1) to (4.5) is locally asymptotically stable if $R_0 > 1$.*

*Proof.* the linearization of the Jacobian matrix of the system of equations (4.1) to (4.5) at any point is

$$J(S, I_1, I_2, S_p, A) = \begin{pmatrix} -a-\mu & -\left(\frac{\beta_1+\sigma}{N}\right)S & -\left(\frac{\beta_2+\sigma}{N}\right)S & 0 & 0 \\ a & \left[\frac{\beta_1+\sigma}{N}\right)S - \Delta_1 & \left(\frac{\beta_2+\sigma}{N}\right)S & 0 & 0 \\ 0 & \theta & -\Delta_2 & 0 & 0 \\ 0 & k_1 & k_2 & -\mu & 0 \\ 0 & \delta_1 & \delta_2 & 0 & -(\alpha+\mu) \end{pmatrix}$$

Where $a = [\beta_1 I_1 + \beta_2 I_2]\frac{1}{N} + \sigma[I_1 + I_2]\frac{1}{N}$

At the endemic equilibrium point the above Jacobian matrix becomes

$$J(E^*) = \begin{bmatrix} -a-\mu & -\left(\frac{\beta_1+\sigma}{N}\right)S^* & -\left(\frac{\beta_2+\sigma}{N}\right)S^* & 0 & 0 \\ a & \left(\frac{\beta_1+\sigma}{N}\right)S^* - \Delta_1 & \left(\frac{\beta_2+\sigma}{N}\right)S^* & 0 & 0 \\ 0 & \theta & -\Delta_2 & 0 & 0 \\ 0 & k_1 & k_2 & -\mu & 0 \\ 0 & \delta_1 & \delta_2 & 0 & -(\alpha+\mu) \end{bmatrix}$$

$$= \begin{bmatrix} -a-\mu & -\left(\frac{\beta_1+\sigma}{R_0}\right) & -\left(\frac{\beta_2+\sigma}{R_0}\right) & 0 & 0 \\ a & \left(\frac{\beta_1+\sigma}{R_0}\right) - \Delta_1 & \left(\frac{\beta_2+\sigma}{R_0}\right) & 0 & 0 \\ 0 & \theta & -\Delta_2 & 0 & 0 \\ 0 & k_1 & k_2 & -\mu & 0 \\ 0 & \delta_1 & \delta_2 & 0 & -(\alpha+\mu) \end{bmatrix}$$

The characteristic equation of the Jacobian matrix is $|J(E^*) - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -a-\mu-\lambda & -\left(\frac{\beta_1+\sigma}{R_0}\right) & -\left(\frac{\beta_2+\sigma}{R_0}\right) & 0 & 0 \\ a & \left(\frac{\beta_1+\sigma}{R_0}\right) - \Delta_1 - \lambda & \left(\frac{\beta_2+\sigma}{R_0}\right) & 0 & 0 \\ 0 & \theta & -\Delta_2-\lambda & 0 & 0 \\ 0 & k_1 & k_2 & -\mu-\lambda & 0 \\ 0 & \delta_1 & \delta_2 & 0 & -(\alpha+\mu)-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\alpha+\mu+\lambda) \begin{vmatrix} -a-\mu-\lambda & -\left(\frac{\beta_1+\sigma}{R_0}\right) & -\left(\frac{\beta_2+\sigma}{R_0}\right) & 0 \\ a & \left(\frac{\beta_1+\sigma}{R_0}\right) - \Delta_1 - \lambda & \left(\frac{\beta_2+\sigma}{R_0}\right) & 0 \\ 0 & \theta & -\Delta_2-\lambda & 0 \\ 0 & k_1 & k_2 & -\mu-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\alpha+\mu+\lambda)(\mu+\lambda) \begin{vmatrix} -a-\mu-\lambda & -\left(\frac{\beta_1+\sigma}{R_0}\right) & -\left(\frac{\beta_2+\sigma}{R_0}\right) \\ a & \left(\frac{\beta_1+\sigma}{R_0}\right) - \Delta_1 - \lambda & \left(\frac{\beta_2+\sigma}{R_0}\right) \\ 0 & \theta & -\Delta_2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\alpha+\mu+\lambda)(\mu+\lambda) \left( (a+\mu+\lambda) \begin{vmatrix} \left(\frac{\beta_1+\sigma}{R_0}\right) - \Delta_1 - \lambda & \left(\frac{\beta_2+\sigma}{R_0}\right) \\ \theta & -\Delta_2-\lambda \end{vmatrix} - a \begin{vmatrix} -\left(\frac{\beta_1+\sigma}{R_0}\right) & -\left(\frac{\beta_2+\sigma}{R_0}\right) \\ \theta & -\Delta_2-\lambda \end{vmatrix} \right) =$$

$0$

$$\Rightarrow (\alpha+\mu+\lambda)(\mu+\lambda)(a+\mu+\lambda)(\Delta_2+\lambda)\left(\Delta_1+\lambda-\left(\frac{\beta_1+\sigma}{R_0}\right)\right)$$
$$-(\alpha+\mu+\lambda)(\mu+\lambda)\left((a+\mu+\lambda)\theta\left(\frac{\beta_2+\sigma}{R_0}\right)+a\left[(\Delta_2+\lambda)\left(\frac{\beta_1+\sigma}{R_0}\right)+\theta\left(\frac{\beta_2+\sigma}{R_0}\right)\right]\right)=0$$
$$\Rightarrow (\alpha+\mu+\lambda)(\mu+\lambda)(a+\mu+\lambda)(\Delta_1\Delta_2+\lambda\Delta_1+\lambda\Delta_2+\lambda^2-\Delta_2\left(\frac{\beta_1+\sigma}{R_0}\right))$$
$$-(\alpha+\mu+\lambda)(\mu+\lambda)(a+\mu+\lambda)\left[\lambda\left(\frac{\beta_1+\sigma}{R_0}\right)+\theta\left(\frac{\beta_2+\sigma}{R_0}\right)\right]$$
$$+(\alpha+\mu+\lambda)(\mu+\lambda)a\left[\Delta_2\left(\frac{\beta_1+\sigma}{R_0}\right)+\lambda\left(\frac{\beta_1+\sigma}{R_0}\right)+\theta\left(\frac{\beta_2+\sigma}{R_0}\right)\right]=0$$
$$\Rightarrow (\alpha+\mu+\lambda)(\mu+\lambda)(a+\mu+\lambda)\left[\Delta_1\Delta_2+\lambda\Delta_1+\lambda\Delta_2+\lambda^2-\Delta_1\Delta_2-\lambda\left(\frac{\beta_1+\sigma}{R_0}\right)\right]$$
$$+(\alpha+\mu+\lambda)(\mu+\lambda)a\left[\lambda\left(\frac{\beta_1+\sigma}{R_0}\right)+\Delta_1\Delta_2\right]=0$$

From this we have

$$(\alpha+\mu+\lambda)=0 \Rightarrow \lambda=-(\alpha+\mu)\ and$$

$$(\mu+\lambda)=0 \Rightarrow \lambda=-\mu$$

For the above two equations we get negative eigenvalues.

For the equation

$(a+\mu+\lambda)\left[\lambda\Delta_1+\lambda\Delta_2+\lambda^2-\lambda\left(\frac{\beta_1+\sigma}{R_0}\right)\right]+a\lambda\left(\frac{\beta_1+\sigma}{R_0}\right)+a\Delta_1\Delta_2=0$

The solution can be obtained as follows:

$$\Rightarrow \lambda a\Delta_1+\lambda a\Delta_2+a\lambda^2-\lambda a\left(\frac{\beta_1+\sigma}{R_0}\right)+\lambda\mu\Delta_1+\lambda\mu\Delta_2+\mu\lambda^2-\lambda\mu\left(\frac{\beta_1+\sigma}{R_0}\right)+\lambda^2\Delta_1$$
$$+\lambda^2\Delta_2+\lambda^3-\lambda^2\left(\frac{\beta_1+\sigma}{R_0}\right)+\lambda a\left(\frac{\beta_1+\sigma}{R_0}\right)+a\Delta_1\Delta_2=0$$

$$\Rightarrow \lambda^3+\lambda^2\left[a+\mu+\Delta_1+\Delta_2-\left(\frac{\beta_1+\sigma}{R_0}\right)\right]$$
$$+\lambda\left[a_1+a\Delta_2-a\left(\frac{\beta_1+\sigma}{R_0}\right)+\mu\Delta_1+\mu\Delta_2-\mu\left(\frac{\beta_1+\sigma}{R_0}\right)+a\left(\frac{\beta_1+\sigma}{R_0}\right)\right]+a\Delta_1\Delta_2=0$$

$$\Rightarrow \lambda^3+\lambda^2\left[a+\mu+\Delta_1+\Delta_2-\left(\frac{\beta_1+\sigma}{R_0}\right)\right]+\lambda\left[a\Delta_1+a\Delta_2+\mu\Delta_1+\mu\Delta_2-\mu\left(\frac{\beta_1+\sigma}{R_0}\right)\right]$$
$$+a\Delta_1\Delta_2=0$$

Then the above characteristic equation is given by

$$P(\lambda)=\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0$$

Where

$A_1=\left[a+\mu+\Delta_1+\Delta_2-\left(\frac{\beta_1+\sigma}{R_0}\right)\right]$

$A_2=\left[a\Delta_1+a\Delta_2+\mu\Delta_1+\mu\Delta_2-\mu\left(\frac{\beta_1+\sigma}{R_0}\right)\right]$

$A_3=a\Delta_1\Delta_2$

Since $R_0=\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}$ we have

$$R_0\Delta_1\Delta_2=\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)$$

$$\Rightarrow R_0\Delta_1\Delta_2>\Delta_2(\beta_1+\sigma)$$

$$\Rightarrow \Delta_1>\frac{\Delta_2(\beta_1+\sigma)}{R_0\Delta_2}$$

$$\Rightarrow \Delta_1>\frac{(\beta_1+\sigma)}{R_0}$$

$$\Rightarrow \Delta_1 - \frac{(\beta_1 + \sigma)}{R_0} > 0$$

Thus we can observe that $A_1, A_2$ and $A_3$ are all positive quantities. Here we can conclude that all coefficients of the characteristic polynomial are positive.

To see the sign of eigenvalues we use Routh – Hurwitz criteria.

Consider the following Routh – Hurwitz array

| | | |
|---|---|---|
| $\lambda^3$ | 1 | $A_2$ |
| $\lambda^2$ | $A_1$ | $A_3$ |
| $\lambda^1$ | B | 0 |
| $\lambda^0$ | C | |

where $B = \frac{-1}{A_1}\begin{vmatrix} 1 & A_2 \\ A_1 & A_3 \end{vmatrix}$ and $C = \frac{-1}{B}\begin{vmatrix} A_1 & A_3 \\ B & 0 \end{vmatrix}$

Now

$$B = \frac{-1}{A_1}\begin{vmatrix} 1 & A_2 \\ A_1 & A_3 \end{vmatrix} = \frac{-1}{A_1}(A_3 - A_1A_2)$$

$$= \frac{1}{A_1}(A_1A_2 - A_3)$$

$= \frac{1}{\left[a+\mu+\Delta_1+\Delta_2-\left(\frac{\beta_1+\sigma}{R_0}\right)\right]}$

$$\left[\left[a+\mu+\Delta_1+\Delta_2-\left(\frac{\beta_1+\sigma}{R_0}\right)\right]\left[a\Delta_1+a\Delta_2+\mu\Delta_1+\mu\Delta_2-\mu\left(\frac{\beta_1+\sigma}{R_0}\right)\right]-a\Delta_1\Delta_2\right]$$

$= \frac{1}{[a+\mu+\Delta_2+K]}\left([a+\mu+\Delta_2+K]\,[a\Delta_1+a\Delta_2+\mu\Delta_2+\mu K]-a\Delta_1\Delta_2\right)$

$= \frac{1}{[a+\mu+\Delta_2+K]}[a^2\Delta_1+a^2\Delta_2+a\mu\Delta_2+a\mu K+a\mu\Delta_1+a\mu\Delta_2+\mu^2\Delta_2+\mu^2K]$

$+\frac{1}{[a+\mu+\Delta_2+K]}[a\Delta_1\Delta_2+a{\Delta_2}^2+\mu{\Delta_2}^2+\mu\Delta_2K+aK\Delta_1+aK\Delta_2+K\mu\Delta_2+\mu K^2-a\Delta_1\Delta_2]$

$= \frac{1}{[a+\mu+\Delta_2+K]}[a^2\Delta_1+a^2\Delta_2+2a\mu\Delta_2+a\mu K+a\mu\Delta_1+\mu^2\Delta_2+\mu^2K]$

$+\frac{1}{[a+\mu+\Delta_2+K]}[a{\Delta_2}^2+\mu{\Delta_2}^2+\mu\Delta_2K+aK\Delta_1+aK\Delta_2+K\mu\Delta_2+\mu K^2] > 0$

Where $K = \Delta_1 - (\frac{\beta_1+\sigma}{R_0} > 0$

$$\Rightarrow B > 0$$

$$C = \frac{-1}{B}\begin{vmatrix} A_1 & A_3 \\ B & 0 \end{vmatrix} = \frac{-1}{B}(0 - BA_3)$$

$$= \frac{BA_3}{B} = A_3 > 0$$

$$\Rightarrow C > 0$$

Since all elements of the first column of the array have the same sign then by Routh – Hurwitz criteria all roots of the characteristic equation have negative real part, thus from above result we can say that the endemic equilibrium point is locally asymptotically stable. □

**Global Stability of Endemic Equilibrium point**

**Theorem 4.6.** *the endemic equilibrium point $E^*$ is globally asymptotically stable if $R_0 > 1$.*

*Proof.* Consider the following Lyapunov function

$V = (S - S^* lnS) + (I_1 - I_1^* lnI_1) + \gamma_1 (I_2 - I_2^* lnI_2) + \gamma_2 \left(S_p - S_p^* lnS_p\right) + \gamma_3 (A - A^* lnA)$

Where $\gamma_i^{'}s$ for $i = 1, 2, 3$ are non-negative quantities.

Differentiating $V$ with respect to time gives

$\frac{dV}{dt} = \left(1 - \frac{S^*}{S}\right) \frac{dS}{dt} + \left(1 - \frac{I_1^*}{I_1}\right) \frac{dI_1}{dt} + \gamma_1 \left(1 - \frac{I_2^*}{I_2}\right) \frac{dI_2}{dt} + \gamma_2 \left(1 - \frac{S_p^*}{S_p}\right) \frac{dS_p}{dt} + \gamma_3 \left(1 - \frac{A^*}{A}\right) \frac{dA}{dt}$

Substituting the expressions for the derivatives in $\frac{dV}{dt}$, it follows that

$$\begin{aligned}\frac{dV}{dt} &= \left(1 - \frac{S^*}{S}\right) \left[Q_0 - \left[\left(\frac{\beta_1+\sigma}{N}\right) I_1 + \left(\frac{\beta_2+\sigma}{N}\right) I_2\right] S - \mu S\right] \\ &\quad + \left(1 - \frac{I_1^*}{I_1}\right) \left[\left[\left(\frac{\beta_1+\sigma}{N}\right) I_1 + \left(\frac{\beta_2+\sigma}{N}\right) I_2\right] S - \Delta_1 I_1\right] + \gamma_1 \left(1 - \frac{I_2^*}{I_2}\right) [\theta I_1 - \Delta_2 I_2] \\ &\quad + \gamma_2 \left(1 - \frac{S_p^*}{S_p}\right) [k_1 I_1 + k_2 I_2 - \mu S_p] + \gamma_3 \left(1 - \frac{A^*}{A}\right) [\delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu) A]\end{aligned}$$

Using the relation $Q_0 = \left[\left(\frac{\beta_1+\sigma}{N}\right) I_1^* + \left(\frac{\beta_2+\sigma}{N}\right) I_2^*\right] S^* + \mu S^*$, we have from the first equation of the system (4.1)−(4.5) at the steady state that $\frac{dV}{dt}$ can be written as

$$\begin{aligned}\frac{dV}{dt} &= \left(1 - \frac{S^*}{S}\right) \left[\left[\left(\frac{\beta_1+\sigma}{N}\right) I_1^* + \left(\frac{\beta_2+\sigma}{N}\right) I_2^*\right] S^* + \mu S^* - \left[\left(\frac{\beta_1+\sigma}{N}\right) I_1 + \left(\frac{\beta_2+\sigma}{N}\right) I_2\right] S - \mu S\right] \\ &\quad + \left(1 - \frac{I_1^*}{I_1}\right) \left[\left[\left(\frac{\beta_1+\sigma}{N}\right) I_1 + \left(\frac{\beta_2+\sigma}{N}\right) I_2\right] S - \Delta_1 I_1\right] + \gamma_1 \left(1 - \frac{I_2^*}{I_2}\right) [\theta I_1 - \Delta_2 I_2] \\ &\quad + \gamma_2 \left(1 - \frac{S_p^*}{S_p}\right) [k_1 I_1 + k_2 I_2 - \mu S_p] + \gamma_3 \left(1 - \frac{A^*}{A}\right) [\delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu) A]\end{aligned}$$

This can then be simplified to

$$\begin{aligned}\frac{dV}{dt} &= \left(1 - \frac{S^*}{S}\right) \left(\frac{\beta_1+\sigma}{N}\right) I_1^* S^* + \left(1 - \frac{S^*}{S}\right) \left(\frac{\beta_2+\sigma}{N}\right) I_2^* S^* + \left(1 - \frac{S^*}{S}\right) (S^* - S) \mu - \left(\frac{\beta_1+\sigma}{N}\right) I_1 S \\ &\quad - \left(\frac{\beta_2+\sigma}{N}\right) I_2 S + \left(\frac{\beta_1+\sigma}{N}\right) I_1 S^* + \left(\frac{\beta_2+\sigma}{N}\right) I_2 S^* + \left(1 - \frac{I_1^*}{I_1}\right) \left[\left[\left(\frac{\beta_1+\sigma}{N}\right) I_1 + \left(\frac{\beta_2+\sigma}{N}\right) I_2\right] S - \Delta_1 I_1\right] \\ &\quad + \gamma_1 \left(1 - \frac{I_2^*}{I_2}\right) [\theta I_1 - \Delta_2 I_2] + \gamma_2 \left(1 - \frac{S_p^*}{S_p}\right) [k_1 I_1 + k_2 I_2 - \mu S_p] + \gamma_3 \left(1 - \frac{A^*}{A}\right) [\delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu) A]\end{aligned}$$

Using the relation at the steady state

$\Delta_1 I_1^* = \left[\left(\frac{\beta_1+\sigma}{N}\right) I_1^* + \left(\frac{\beta_2+\sigma}{N}\right) I_2^*\right] S^*, \quad \Delta_2 I_2^* = \theta I_1^*, \ \mu S_p^* = k_1 I_1^* + k_2 I_2^*, \ \mu S_p^* = k_1 I_1^* + k_2 I_2^*,$

$(\alpha + \mu) A = \delta_1 I_1^* + \delta_2 I_2^*$

We again simplify

$$\begin{aligned}\frac{dV}{dt} &= \mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \left(1 - \frac{S^*}{S}\right) \left(\frac{\beta_1+\sigma}{N}\right) I_1^* S^* + \left(1 - \frac{S^*}{S}\right) \left(\frac{\beta_2+\sigma}{N}\right) I_2^* S^* - \left(\frac{\beta_1+\sigma}{N}\right) I_1 S \\ &\quad - \left(\frac{\beta_2+\sigma}{N}\right) I_2 S + \left(\frac{\beta_1+\sigma}{N}\right) I_1 S^* + \left(\frac{\beta_2+\sigma}{N}\right) I_2 S^* + \left(\frac{\beta_1+\sigma}{N}\right) I_1 S + \left(\frac{\beta_2+\sigma}{N}\right) I_2 S \\ &\quad - \left(\frac{\beta_1+\sigma}{N}\right) I_1^* S - \left(\frac{\beta_2+\sigma}{N}\right) \frac{I_1^*}{I_1} I_2 S - \left(1 - \frac{I_1^*}{I_1}\right) \Delta_1 I_1 + \gamma_1 \left(1 - \frac{I_2^*}{I_2}\right) [\theta I_1 - \Delta_2 I_2] \\ &\quad + \gamma_2 \left(1 - \frac{S_p^*}{S_p}\right) [k_1 I_1 + k_2 I_2 - \mu S_p] + \gamma_3 \left(1 - \frac{A^*}{A}\right) [\delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu) A] \\ &= \mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \left(1 - \frac{S^*}{S} - \frac{S}{S^*}\right) \left(\frac{\beta_1+\sigma}{N}\right) I_1^* S^* + \left(1 - \frac{S^*}{S}\right) \left(\frac{\beta_2+\sigma}{N}\right) I_2^* S^* + \left(\frac{\beta_1+\sigma}{N}\right) I_1 S^* \\ &\quad + \left(\frac{\beta_2+\sigma}{N}\right) I_2 S^* - \left(\frac{\beta_1+\sigma}{N}\right) I_1^* S - \left(\frac{\beta_2+\sigma}{N}\right) \frac{I_1^*}{I_1} I_2 S - \left(1 - \frac{I_1^*}{I_1}\right) \left[\left(\frac{\beta_1+\sigma}{N}\right) I_1^* S^* + \left(\frac{\beta_2+\sigma}{N}\right) I_2^* S^*\right] \frac{I_1}{I_1^*} \\ &\quad + \gamma_1 \left(1 - \frac{I_2^*}{I_2}\right) [\theta I_1 - \Delta_2 I_2] + \gamma_2 \left(1 - \frac{S_p^*}{S_p}\right) [k_1 I_1 + k_2 I_2 - \mu S_p] + \gamma_3 \left(1 - \frac{A^*}{A}\right) [\delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu) A] \\ &= \mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \left(1 - \frac{S^*}{S} - \frac{S}{S^*}\right) \left(\frac{\beta_1+\sigma}{N}\right) I_1^* S^* + \left(1 - \frac{S^*}{S}\right) \left(\frac{\beta_2+\sigma}{N}\right) I_2^* S^* + \left(\frac{\beta_1+\sigma}{N}\right) I_1 S^*\end{aligned}$$

$$
\begin{aligned}
&+\left(\tfrac{\beta_2+\sigma}{N}\right) I_2S^* - \left(\tfrac{\beta_1+\sigma}{N}\right) I_1^*S - \left(\tfrac{\beta_2+\sigma}{N}\right) \tfrac{I_1^*}{I_1} I_2S - \left(\tfrac{\beta_1+\sigma}{N}\right) I_1S^* \\
&- \left(\tfrac{\beta_2+\sigma}{N}\right) I_2^*S^* \tfrac{I_1}{I_1^*} + \left(\tfrac{\beta_1+\sigma}{N}\right) I_1^*S^* + \left(\tfrac{\beta_2+\sigma}{N}\right) I_2^*S^* + \gamma_1 \left(1 - \tfrac{I_2^*}{I_2}\right) [\theta I_1 - \Delta_2 I_2] \\
&+\gamma_2 \left(1 - \tfrac{S_p^*}{S_p}\right) [k_1 I_1 + k_2 I_2 - \mu S_p] + \gamma_3 \left(1 - \tfrac{A^*}{A}\right) [\delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu) A] \\
&= \mu S^* \left(2 - \tfrac{S}{S^*} - \tfrac{S^*}{S}\right) + \left(\tfrac{\beta_1+\sigma}{N}\right) I_1^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) + \left(\tfrac{\beta_2+\sigma}{N}\right) I_2^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{I_1}{I_1^*}\right) \\
&+ \left(\tfrac{\beta_2+\sigma}{N}\right) I_2S^* - \left(\tfrac{\beta_1+\sigma}{N}\right) I_1^*S - \left(\tfrac{\beta_2+\sigma}{N}\right) \tfrac{I_1^*}{I_1} I_2S + \gamma_1 \left(1 - \tfrac{I_2^*}{I_2}\right) [\theta I_1 - \Delta_2 I_2] \\
&+\gamma_2 \left(1 - \tfrac{S_p^*}{S_p}\right) [k_1 I_1 + k_2 I_2 - \mu S_p] + \gamma_3 \left(1 - \tfrac{A^*}{A}\right) [\delta_1 I_1 + \delta_2 I_2 - (\alpha + \mu) A] \\
&= \mu S^* \left(2 - \tfrac{S}{S^*} - \tfrac{S^*}{S}\right) + \left(\tfrac{\beta_1+\sigma}{N}\right) I_1^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) + \left(\tfrac{\beta_2+\sigma}{N}\right) I_2^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{I_1}{I_1^*}\right) \\
&+ \left(\tfrac{\beta_2+\sigma}{N}\right) I_2S^* - \left(\tfrac{\beta_2+\sigma}{N}\right) \tfrac{I_1^*}{I_1} I_2S + \gamma_1 \left(1 - \tfrac{I_2^*}{I_2}\right) \theta I_1 - \gamma_1 \left(1 - \tfrac{I_2^*}{I_2}\right) \Delta_2 I_2 \\
&+\gamma_2 \left(1 - \tfrac{S_p^*}{S_p}\right) k_1 I_1 + \gamma_2 \left(1 - \tfrac{S_p^*}{S_p}\right) k_2 I_2 - \gamma_2 \left(1 - \tfrac{S_p^*}{S_p}\right) \mu S_p + \gamma_3 \left(1 - \tfrac{A^*}{A}\right) \delta_1 I_1 \\
&+\gamma_3 \left(1 - \tfrac{A^*}{A}\right) \delta_2 I_2 - \gamma_3 \left(1 - \tfrac{A^*}{A}\right) (\alpha + \mu) A \\
&= \mu S^* \left(2 - \tfrac{S}{S^*} - \tfrac{S^*}{S}\right) + \left(\tfrac{\beta_1+\sigma}{N}\right) I_1^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) + \left(\tfrac{\beta_2+\sigma}{N}\right) I_2^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{I_1}{I_1^*}\right) \\
&+ \left(\tfrac{\beta_2+\sigma}{N}\right) I_2S^* - \left(\tfrac{\beta_2+\sigma}{N}\right) \tfrac{I_1^*}{I_1} I_2S + \gamma_1\theta I_1 - \gamma_1\theta I_1 \tfrac{I_2^*}{I_2} - \gamma_1\Delta_2 I_2 + \gamma_1\Delta_2 I_2^* + \gamma_2 k_1 I_1 \\
&-\gamma_2 k_1 \tfrac{S_p^*}{S_p} I_1 + \gamma_2 k_2 I_2 - \gamma_2 k_2 \tfrac{S_p^*}{S_p} I_2 - \gamma_2 k_1 I_1^* \tfrac{S_p}{S_p^*} + \gamma_2 k_2 I_2^* \tfrac{S_p}{S_p^*} + \gamma_2 k_1 I_1^* + \gamma_2 k_2 I_2^* + \gamma_3 \delta_1 I_1 \\
&-\gamma_3 \delta_1 I_1 \tfrac{A^*}{A} + \gamma_3 \delta_2 I_2 - \gamma_3 \delta_2 I_2 \tfrac{A^*}{A} - \gamma_3 \delta_1 I_1^* \tfrac{A}{A^*} - \gamma_3 \delta_2 I_2^* \tfrac{A}{A^*} + \gamma_3 \delta_1 I_1^* + \gamma_3 \delta_2 I_2^* \\
&= \mu S^* \left(2 - \tfrac{S}{S^*} - \tfrac{S^*}{S}\right) + \left(\tfrac{\beta_1+\sigma}{N}\right) I_1^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) + \left(\tfrac{\beta_2+\sigma}{N}\right) I_2^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{I_1}{I_1^*}\right) \\
&+ \left(\tfrac{\beta_2+\sigma}{N}\right) I_2S^* - \left(\tfrac{\beta_2+\sigma}{N}\right) \tfrac{I_1^*}{I_1} I_2S + \gamma_1\theta I_1 \left(1 - \tfrac{I_2^*}{I_2}\right) + \gamma_1\Delta_2 I_2^* \left(1 - \tfrac{I_2}{I_2^*}\right) + \gamma_2 k_1 I_1 \\
&-\gamma_2 k_1 \tfrac{S_p^*}{S_p} I_1 - \gamma_2 k_1 I_1^* \tfrac{S_p}{S_p^*} + \gamma_2 k_1 I_1^* + \gamma_2 k_2 I_2 - \gamma_2 k_2 \tfrac{S_p^*}{S_p} I_2 + \gamma_2 k_2 I_2^* \tfrac{S_p}{S_p^*} + \gamma_2 k_2 I_2^* \\
&+\gamma_3 \delta_1 I_1 - \gamma_3 \delta_1 I_1 \tfrac{A^*}{A} - \gamma_3 \delta_1 I_1^* \tfrac{A}{A^*} + \gamma_3 \delta_1 I_1^* + \gamma_3 \delta_2 I_2 - \gamma_3 \delta_2 I_2 \tfrac{A^*}{A} - \gamma_3 \delta_2 I_2^* \tfrac{A}{A^*} + \gamma_3 \delta_2 I_2^* \\
&= \mu S^* \left(2 - \tfrac{S}{S^*} - \tfrac{S^*}{S}\right) + \left(\tfrac{\beta_1+\sigma}{N}\right) I_1^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) + \left(\tfrac{\beta_2+\sigma}{N}\right) I_2^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{I_1}{I_1^*}\right) \\
&+ \left(\tfrac{\beta_2+\sigma}{N}\right) I_2S^* - \left(\tfrac{\beta_2+\sigma}{N}\right) \tfrac{I_1^*}{I_1} I_2S + \gamma_1\theta I_1 \left(1 - \tfrac{I_2^*}{I_2}\right) + \gamma_1\theta I_1^* \left(1 - \tfrac{I_2}{I_2^*}\right) + \gamma_2 k_1 I_1 \\
&+\gamma_2 k_1 I_1^* \left(2 - \tfrac{S_p}{S_p^*} - \tfrac{S_p^* I_1}{S_p I_1^*}\right) - \gamma_2 k_1 I_1^* + \gamma_2 k_2 I_2 + \gamma_2 k_2 I_2^* \left(2 - \tfrac{S_p}{S_p^*} - \tfrac{S_p^* I_2}{S_p I_2^*}\right) - \gamma_2 k_2 I_2^* + \gamma_3 \delta_1 I_1 \\
&+\gamma_3 \delta_1 I_1^* \left(2 - \tfrac{A}{A^*} - \tfrac{A^* I_1}{A I_1^*}\right) - \gamma_3 \delta_1 I_1^* + \gamma_3 \delta_2 I_2 + \gamma_3 \delta_2 I_2^* \left(2 - \tfrac{A}{A^*} - \tfrac{A^* I_2}{A I_2^*}\right) - \gamma_3 \delta_2 I_2^* \\
&= \mu S^* \left(2 - \tfrac{S}{S^*} - \tfrac{S^*}{S}\right) + \left(\tfrac{\beta_1+\sigma}{N}\right) I_1^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) + \left(\tfrac{\beta_2+\sigma}{N}\right) I_2^*S^* \left(2 - \tfrac{S^*}{S} - \tfrac{I_1}{I_1^*}\right) + \left(\tfrac{\beta_2+\sigma}{N}\right) S^*I_2 \\
&- \left(\tfrac{\beta_2+\sigma}{N}\right) \tfrac{I_1^*}{I_1} I_2S + \gamma_1\theta I_1^* \left(1 - \tfrac{I_2^*}{I_2^*} \tfrac{I_1}{I_1^*}\right) - \gamma_1\theta I_1^* \tfrac{I_2}{I_2^*} + \gamma_1\theta I_1 + \gamma_2 k_1 I_1 + \gamma_2 k_1 I_1^* \left(2 - \tfrac{S_p}{S_p^*} - \tfrac{S_p^* I_1}{S_p I_1^*}\right) \\
&-\gamma_2 k_1 I_1^* + \gamma_2 k_2 I_2^* \left(2 - \tfrac{S_p}{S_p^*} - \tfrac{S_p^* I_2}{S_p I_2^*}\right) - \gamma_2 k_2 I_2^* + \gamma_3 \delta_1 I_1 + \gamma_3 \delta_1 I_1^* \left(2 - \tfrac{A}{A^*} - \tfrac{A^* I_1}{A I_1^*}\right) \\
&-\gamma_3 \delta_1 I_1^* + \gamma_3 \delta_2 I_2 + \gamma_2 k_2 I_2 + \gamma_3 \delta_2 I_2^* \left(2 - \tfrac{A}{A^*} - \tfrac{A^* I_2}{A I_2^*}\right) - \gamma_3 \delta_2 I_2^*
\end{aligned}
$$

The coefficient $\gamma_1$ is obtained from the expression $\left(\frac{\beta_2+\sigma}{N}\right) S^* - \gamma_1\theta \frac{I_1^*}{I_2^*} = 0$

$$\Rightarrow \gamma_1\theta \frac{I_1^*}{I_2^*} = \left(\frac{\beta_2+\sigma}{N}\right) S^*$$

$$\Rightarrow \gamma_1 = \left(\frac{\beta_2+\sigma}{N}\right) \frac{S^*I_2^*}{\theta I_1^*}$$

And the coefficient $\gamma_2$ and $\gamma_3$ are obtained from the expression $\gamma_3\delta_2 + \gamma_2 k_2 = 0$

$\Rightarrow \gamma_2 = \gamma_3 = 0$ since $\gamma_i's$ for $i = 1, 2, 3$ are non-negative quantities.

Thus

$\frac{dV}{dt} = \mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \left(\frac{\beta_1+\sigma}{N}\right) I_1^* S^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + \left(\frac{\beta_2+\sigma}{N}\right) I_2^* S^* \left(2 - \frac{S^*}{S} - \frac{I_1}{I_1^*}\right)$
$- \left(\frac{\beta_2+\sigma}{N}\right) \frac{I_1^*}{I_1} I_2 S + \left(\frac{\beta_2+\sigma}{N}\right) S^* I_2^* \left(1 - \frac{I_2^* I_1}{I_2 I_1^*}\right) + \left(\frac{\beta_2+\sigma}{N}\right) \frac{S^* I_2^*}{I_1^*} I_1$
$= \mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \left(\frac{\beta_1+\sigma}{N}\right) I_1^* S^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + \left(\frac{\beta_2+\sigma}{N}\right) I_2^* S^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right)$
$- \left(\frac{\beta_2+\sigma}{N}\right) \frac{I_1^*}{I_1} I_2 S + \left(\frac{\beta_2+\sigma}{N}\right) S^* I_2^* \left(1 - \frac{I_2^* I_1}{I_2 I_1^*} - \frac{I_1^* I_2}{I_1 I_2^*}\right) + \left(\frac{\beta_2+\sigma}{N}\right) S^* \frac{I_1^* I_2}{I_1} + \left(\frac{\beta_2+\sigma}{N}\right) I_2^* S$

Note that $\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right)$, $\left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right)$ and $\left(1 - \frac{I_2^* I_1}{I_2 I_1^*} - \frac{I_1^* I_2}{I_1 I_2^*}\right)$ are less than or equal to zero by arithmetic mean -geometric mean inequality.

This gives

$$\frac{dV}{dt} = Z - Y$$

Where

$$Z = \left(\frac{\beta_2 + \sigma}{N}\right) S^* \frac{I_1^* I_2}{I_1} + \left(\frac{\beta_2 + \sigma}{N}\right) I_2^* S$$

and

$Y = -\left[\mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \left(\frac{\beta_1+\sigma}{N}\right) I_1^* S^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + \left(\frac{\beta_2+\sigma}{N}\right) I_2^* S^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right)\right]$
$+ \left(\frac{\beta_2+\sigma}{N}\right) \frac{I_1^*}{I_1} I_2 S - \left(\frac{\beta_2+\sigma}{N}\right) S^* I_2^* \left(1 - \frac{I_2^* I_1}{I_2 I_1^*} - \frac{I_1^* I_2}{I_1 I_2^*}\right)$

Hence, if $Z < Y$ then, $\frac{dV}{dt}$ will be negative definite, implying that $\frac{dV}{dt} < 0$. Also $\frac{dV}{dt} = 0$, if and only if $S = S^*, I_1 = I_1^*, I_2 = I_2^*, S_p = S_p^*$ and $A = A^*$.

Therefore, the largest compact invariant set in $\{(S^*, I_1^*, I_2^*, S_p^*, A^*) \epsilon \Omega : \frac{dV}{dt} = 0\}$ is the singleton $\{E^*\}$, where $E^*$ is endemic equilibrium of the system (4.1)−(4.5). By LaSalle's invariant principle, it then implies that $E^*$ is globally asymptotically stable in $\Omega$ if $Z < Y$. □

# Chapter 5

# The Dynamics of HIV/AIDS with Age Structure and Inflow Immigrants in Ethiopia

## 5.1 Introduction

HIV/AIDS remains a major global health problem affecting approximately 70 million people worldwide causing significant morbidity and mortality [114].

In Ethiopia in 2018, 690 000 people were living with HIV. HIV incidence per 1000 uninfected the number of new HIV infections among the uninfected population over one year among all people of all ages was 0.24. 23 000 people were newly infected with HIV and 11 000 people died from an AIDS-related illness. There has been progress in the number of AIDS-related deaths since 2010, with a 45% decrease, from 20 000 deaths to 11 000 deaths. The number of new HIV infections has also decreased, from 29 000 to 23 000 in the same period. The 90-90-90 targets envision that, by 2020, 90% of people living with HIV will know their HIV status, 90% of people who know their HIV-positive status will be accessing treatment and 90% of people on treatment will have suppressed viral loads. In terms of all people living with HIV, reaching the 90-90-90 targets means that 81% of all people living with HIV are on treatment and 73% of all people living with HIV are virally suppressed. In 2018 in Ethiopia 79% of people living with HIV knew their status and 65% of people living with HIV were on treatment [49].

The Human Immunodeficiency Virus (HIV) infects cells of immune system such as helper T cells

(specifically CD4+ T cells), macrophages, and dendritic cells. HIV compromises the human immune system and reduces the ability of the body to fight back infections and diseases. The most advanced stages of HIV infection is usually called Acquired Immunodeficiency Syndrome (AIDS). AIDS is one of the leading causes of death worldwide that is affecting virtually every nation. Even if HIV/AIDS is not permanently curable, main methods used to fight against it are preventive mechanisms (which include: abstinence, faithfulness and protection) which mainly rely on the level of behavioral change of the population, and providing Antiretroviral Therapy (ART) for those infected [105].

Mathematical models have played a major role in increasing our understanding of the dynamics of infectious diseases. Several models have been proposed to study the effects of some factors on the transmission dynamics of these infectious diseases including HIV/AIDS and to provide guidelines as to how the spread can be controlled. Among these models include those of Anderson et al. [90] who presented a preliminary study of the transmission dynamics of HIV by proposing a model to study the effects of various factors on the transmission of the disease, Stilianakis et al. [76] who proposed and gave a detailed analysis of a dynamical model that describes the pathogenesis of HIV, and Tripathi et al. [3] who proposed a model to study the effects of screening of unaware infective on the transmission dynamics of HIV/AIDS. Several other models proposed to study dynamics of HIV/AIDS can be found in ([[1], [11], [12], [21], [56], [59], [69], [71], [84]], and the references therein).

To assess the impact of incidence functions in the estimation of the long-term dynamics of the disease, mathematical models of infectious diseases are useful tools for comparing control strategies and identifying key disease drivers as well as important areas of uncertainty that may be prioritized for urgent research. Large amount of work done on modeling the spread of HIV has been largely restricted to ordinary differential equations, though studies which have incorporated the combination of condom use, public health education campaigns, and treatment of infected individuals for eradication of the epidemic [[57]− [117]].

Mathematical modeling enjoys popularity in both preventing and controlling infectious diseases such as severe acute respiratory syndrome (SARS) [77], human immunodeficiency virus infection/acquired immune deficiency syndrome (HIV/AIDS) [15], H5N1 (avian flu) [112] and H1N1

(swine flu) [87]. In recent years, a lot of efforts have been made to develop realistic diseases and further study the asymptotic behavior of such epidemic models [119]. In the field of studying epidemic model behavior, one of the most important parts is to analyze steady states together with their stability [64].

This thesis considered a deterministic mathematical model for age structure, the combined effect of unaware immigrants, different mode of transmissions and aware immigrants, for predicting the epidemiological trends of HIV that exploits HIV surveillance data to model the disease evolution in Ethiopia. Based on real data collected from different health sectors we recommend some solutions which depend on our control parameters that help the stake holders to control the spread of the considered disease.

## 5.2 Mathematical Model

### 5.2.1 Model Assumption

In the original model [117] they developed a mathematical model to describe the population dynamics of HIV/AIDS disease based on the following assumptions:

- The population under study is heterogeneous and varying with time.
- The whole human population is divided in to five classes.
- The HIV can be transmitted by the sexual intercourse with infective peoples, vertical transmission and blood borne transmission.
- The full blown AIDS class is sexually inactive.
- Assumed that the seropositive/treatment class could not transmit the disease.
- All the new infected people are assumed to be initially unaware of the infection.
- The probability of transferring the disease to susceptible population by unaware infected person is more than by aware infected person i. e. $\beta_1 > \beta_2$.

Here in the present study we develop a mathematical model by incorporating the following assumptions to describe the population dynamics of HIV/AIDS disease:

- Susceptible people divided in to two; sexually mature and sexually immature.
- Sexually mature people infected by sexual contact and blood borne transmissions.
- Sexually immature people infected by blood borne transmission only.
- Pre-AIDS compartment is added and people from this compartment can infect susceptible individuals with sexual contact rate $\beta_3$ and blood borne transmission rate $\sigma$.

### 5.2.2 Compartments of population for the present model

We divided the total population N(t), into seven compartments: $S_1(t)$,$S_2(t)$,$I_1(t)$,$I_2(t)$,$P(t)$,$S_p(t)$, and $A(t)$. $S_1(t)$ represents the number of sexually mature susceptible individuals (age 15 years and above); $S_2(t)$ represents the number of sexually immature susceptible individuals (age below 15 years); $I_1(t)$ represents the number of unaware HIV-positive individuals ; $I_2(t)$ represents the number of aware HIV-positive individuals ; $P(t)$ represents the number of HIV-positive individuals in the pre-AIDS stage; $S_p(t)$ represents the number of individuals who are receiving ART and keeping themselves from unsafe sex and behavioral change; $A(t)$ represents the number of individuals with full-blown AIDS.

### 5.2.3 Flow of the people among the compartments

The sexually mature susceptible individuals are recruited into the population at a constant rate $Q_1$. This sub population is reduced by infection, following effective contact with infected individuals through sexual and blood borne at the rate $[\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N}$ and $\sigma[I_1 + I_2 + P]\frac{S_1}{N}$ respectively. It is reduced further by natural death at a rate $\mu$. The sexually immature susceptible individuals (age below 15 years) are recruited into the population at a constant rate $Q_2$. This sub population is reduced by infection, following effective contact with infected individuals through blood borne only at the rate $\sigma[I_1 + I_2 + P]\frac{S_2}{N}$ . It is reduced further by natural death at a rate $\mu$. Once an individual is infected, he/she becomes infectious. The population of HIV positive individuals or infective who do not know their status is increased by infection of susceptible individuals at the rate $[\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N}$, $\sigma[I_1 + I_2 + P]\frac{S_1}{N}$ and

$\sigma[I_1+I_2+P]\frac{S_2}{N}$, vertical transmission at the rate of $(1-\epsilon)\phi$, variable inflow of unaware infective at the rate of $p_1$ . It is decreased by screening which leads to status awareness at the rate $\theta$, development to Pre-AIDS, progression to full blown AIDS, transform through treatment and natural death at the rates , $u_1, \delta_1, k_1$ and $\mu$, respectively. The population of HIV positives or infective aware of their HIV status is generated by screening of unaware infective at the rate $\theta$, variable inflow of aware infective at the rate of $p_2$ and decreased by development of pre-AIDS at the rate $u_2$, development of full blown AIDS at the rate $\delta_2$, transform through treatment at the rate of $k_2$, and natural death at the rate $\mu$. The pre-AIDS population is increased by the rate $u_1$ from unaware infective and $u_2$ from aware infective. This sub population is diminished by progression to full blown AIDS at the rate $\delta_3$, transform through treatment $k_3$ and natural death at the rate $\mu$, Treated population is increased by the rate of $k_1$ from unaware infective, $I_1, k_2$ from aware infective, $I_2$ and $k_3$ from pre-AIDS, $P$. It is decreased by natural death at the rate $\mu$. Finally, the population of individuals with AIDS is increased by progression to full blown AIDS at the rate$\delta_1$ for unaware infective, $\delta_2$ for aware infective, $\delta_3$ for pre-AIDS individuals. It is decreased by natural death at the rate $\mu$ and by disease-induced mortality at the rate $\alpha$.

### 5.2.4 Flow Diagram of the Model

Using the above assumptions we developed the following flow diagram.

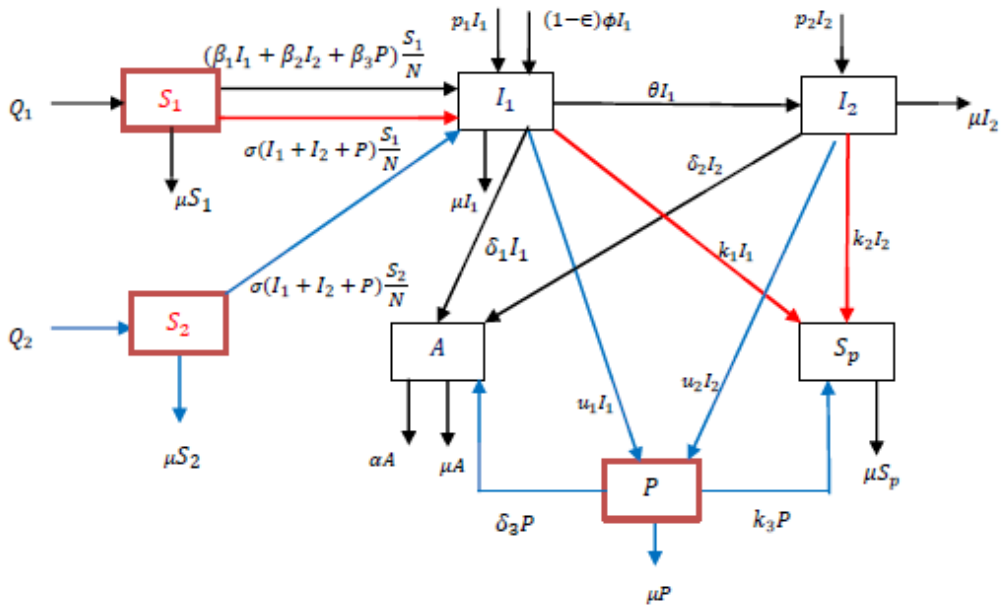


Figure 5.1: Flow diagram of the present model.

Based on the above basic assumptions and flow diagram we do have the following corresponding dynamical system represented by seven non-linear ordinary differential equations.

$$\frac{dS_1}{dt} = Q_1 - [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N} - [I_1 + I_2 + P]\sigma\frac{S_1}{N} - \mu S_1 \tag{5.1}$$

$$\frac{dS_2}{dt} = Q_2 - [I_1 + I_2 + P]\sigma\frac{S_2}{N} - \mu S_2 \tag{5.2}$$

$$\frac{dI_1}{dt} = [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N} + [I_1 + I_2 + P]\sigma\frac{S_1}{N} + [I_1 + I_2 + P]\sigma\frac{S_2}{N} + p_1 I_1 + (1-\epsilon)I_1\phi - (k_1 + \theta + \delta_1 + \mu + u_1)I_1 \tag{5.3}$$

$$\frac{dI_2}{dt} = p_2 I_2 + I_1\theta - (k_2 + \delta_2 + \mu + u_2)I_2 \tag{5.4}$$

$$\frac{dP}{dt} = u_1 I_1 + u_2 I_2 - (\delta_3 + k_3 + \mu)P \tag{5.5}$$

$$\frac{dS_p}{dt} = k_1 I_1 + k_2 I_2 + k_3 P - \mu S_p \tag{5.6}$$

$$\frac{dA}{dt} = \delta_1 I_1 + \delta_2 I_2 + \delta_3 P - (\alpha + \mu)A \tag{5.7}$$

With initial conditions

$$S_1(0) = S_{10}, S_2(0) = S_{20}, I_1(0) = I_{10}, I_2(0) = I_{20}, P(0) = P_0, S_p(0) = S_{p0} and A(0) = A_0 \tag{5.8}$$

Table 5.1: State variables of the HIV/AIDS basic model

| **Symbole** | **Descripition** |
|---|---|
| $S_1(t)$ | Sexually immature Susceptible at time $t$ |
| $S_2(t)$ | Sexually mature Susceptible at time $t$ |
| $I_1(t)$ | HIV-infected individuals unaware of their status |
| $I_2(t)$ | HIV-infected individuals aware of their status |
| $P(t)$ | Pre-AIDS individuals at time $t$ |
| $S_p(t)$ | Treated individuals at time $t$ |
| $A(t)$ | AIDS individuals at time $t$ |

Table 5.2: Parameters of the HIV/AIDS basic model

| **Parameter** | **Parameter Description** |
|---|---|
| $Q_1$ | Recruitment in to sexual mature population |
| $Q_2$ | Recruitment in to sexual immature population |
| $\beta_1$ | The horizontal transmission rate of unaware infective to susceptible individuals |
| $\beta_2$ | The horizontal transmission rate of aware infective to susceptible individuals |
| $\beta_3$ | The horizontal transmission rate of Pre-AIDS to susceptible individuals |
| $\sigma$ | Rate of transmission through blood borne |
| $\delta_1$ | Rate at which unaware infective develop full blown AIDS |
| $\delta_2$ | Rate at which aware infective develop full blown AIDS |
| $\delta_3$ | Progression rate of Pre-AIDS individuals to full blown AIDS |
| $\mu$ | Natural mortality |
| $\theta$ | Rate of status awareness due to screening method |
| $k_1$ | Rate of treatment of unaware infective |
| $k_2$ | Rate of treatment of aware infective |
| $k_3$ | Rate of treatment of Pre-AIDS |
| $p_1$ | Rate of unaware infective immigrants |
| $p_2$ | Rate of aware infective immigrants |

| | |
|---|---|
| $u_1$ | Rate of progress to Pre-AIDS from unaware infective |
| $u_2$ | Rate of progress to Pre-AIDS from aware infective |
| $\phi$ | Rate of vertical transmission |
| $\epsilon$ | Probability of death at birth |
| $\alpha$ | AIDS induced death rate |

### 5.2.5 Model properties

System 5.1− 5.7 will be analyzed in a domain $\Omega \subset R_+^7$ where $\Omega = \{(S_1, S_2, I_1, I_2, P, S_p, A) \in R_+^7\}$.

**Theorem 5.1** (positivity)**.** *The solutions of the dynamical system (5.1− 5.7) with initial conditions (5.8) satisfy* $S_1(t) > 0, S_2(t) > 0, I_1(t) > 0, I_2(t) > 0, S_p(t) > 0, P(t) > 0, A(t) > 0$ *for all* $t > 0$*. The region* $\Omega \subset R_+^7$ *is positively invariant and attracting with respect to system (5.1− 5.7).*

*Proof.* To show the positivity of the solution of the dynamical system comprising the equations 5.1− 5.7, we have to consider and verify each differential equation and show that their solution is positive.

We define:

$\bar{t} = sup\{t > 0 : S_1(t) > 0, S_2(t) > 0, I_1(t) > 0, I_2(t) > 0, P(t) > 0, S_p(t) > 0 \; and \; A(t) > 0\}$

From the continuity of $S_1(t) > 0, S_2(t) > 0, I_1(t) > 0, I_2(t) > 0, P(t) > 0, S_p(t) > 0$ *and* $A(t) > 0$ , we deduce that $\bar{t} > 0$. Now if $\bar{t} = +\infty$, then the claim holds. That is, $S_1(t) > 0, S_2(t) > 0, I_1(t) > 0, I_2(t) > 0, P(t) > 0, S_p(t) > 0$ *and* $A(t) > 0$ for all $t > 0$. But if $0 < \bar{t} < +\infty$, from the definition of $\bar{t}$ it follows that,

$S_1(\bar{t}) = 0$ or $S_2(\bar{t}) = 0$ or $I_1(\bar{t}) = 0$ or $I_2(\bar{t}) = 0$ or $P(\bar{t}) = 0$ or $S_p(\bar{t}) = 0$ or $A(\bar{t})$=0

First let us consider the differential equation (5.1) of the dynamical system

$\frac{dS_1}{dt} = Q_1 - [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P] \frac{S_1}{N} - \sigma [I_1 + I_2 + P] \frac{S_1}{N} - \mu S_1$

$\Rightarrow \frac{dS_1}{dt} + [q + \mu] S_1 = Q_1 \quad$ where $\; q(t) = [(\beta_1 + \sigma) I_1(t) + (\beta_2 + \sigma) I_2(t) + (\beta_3 + \sigma) P(t)] \frac{1}{N(t)}$

This is a first order linear ordinary differential equation.

Now we can find the integrating factor

$\mu_1(t) = e^{\int [q+\mu] dt} = e^{Q(t)+\mu t}$ where $Q(t)$ is the anti-derivative of $q(t)$.

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{Q(t)+\mu t}\frac{dS_1}{dt}+[q+\mu]\,e^{Q(t)+\mu t}S_1=Q_1e^{Q(t)+\mu t}$

$\Rightarrow\left(e^{Q(t)+\mu t}S_1\left(t\right)\right)'=Q_1e^{Q(t)+\mu t}$

Integrating both sides

$\int\limits_0^t\left(e^{Q(s)+\mu s}S_1\left(s\right)\right)'ds=\int\limits_0^tQ_1e^{Q(s)+\mu s}ds$

$\Rightarrow\ e^{Q(s)+\mu s}S_1\left(s\right)\Big|_0^t=\int\limits_0^tQ_1e^{Q(s)+\mu s}ds$

$\Rightarrow e^{Q(t)+\mu t}S_1\left(t\right)-e^{Q(0)}S_1\left(0\right)=\int\limits_0^tQ_1e^{Q(s)+\mu s}ds$

$\Rightarrow e^{Q(t)+\mu t}S_1\left(t\right)=e^{Q(0)}S_1\left(0\right)+\int\limits_0^tQ_1e^{Q(s)+\mu s}ds$

$\Rightarrow S_1\left(t\right)=\frac{e^{Q(0)}}{e^{Q(t)+\mu t}}S_1\left(0\right)+\frac{1}{e^{Q(t)+\mu t}}\int\limits_0^tQ_1e^{Q(s)+\mu s}ds$

$\Rightarrow S_1\left(t\right)=S_1\left(0\right)e^{-Q(t)+Q(0)-\mu t}+\int\limits_0^tQ_1e^{(Q(s)-Q(t))+\mu(s-t)}ds$

From this solution that $S_1(t)$ is positive since $S_1(0)>0, Q_1>0$ and the exponential function always positive.

Secondly let us consider the differential equation (5.2).

$\frac{dS_2}{dt}=Q_2-\sigma\left[I_1+I_2+P\right]\frac{S_2}{N}-\mu S_2$

$\Rightarrow\frac{dS_2}{dt}+\left[q+\mu\right]S_2=Q_2\quad$ where $\ q\left(t\right)=\left[I_1\left(t\right)+I_2\left(t\right)+P\left(t\right)\right]\frac{\sigma}{N(t)}$.

This is a first order linear ordinary differential equation.

Now we can find the integrating factor

$\mu_1\left(t\right)=e^{\int[q+\mu]dt}=e^{Q(t)+\mu t}$ where $Q(t)$ is the anti-derivative of $q(t)$.

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{Q(t)+\mu t}\frac{dS_2}{dt}+[q+\mu]\,e^{Q(t)+\mu t}S_2=Q_2e^{Q(t)+\mu t}$

$\Rightarrow\left(e^{Q(t)+\mu t}S_2\left(t\right)\right)'=Q_2e^{Q(t)+\mu t}$

Integrating both sides

$\int\limits_0^t\left(e^{Q(s)+\mu s}S_2\left(s\right)\right)'ds=\int\limits_0^tQ_2e^{Q(s)+\mu s}ds$

$\Rightarrow\ e^{Q(s)+\mu s}S_2\left(s\right)\Big|_0^t=\int\limits_0^tQ_2e^{Q(s)+\mu s}ds$

$\Rightarrow e^{Q(t)+\mu t}S_2\left(t\right)-e^{Q(0)}S_2\left(0\right)=\int\limits_0^tQ_2e^{Q(s)+\mu s}ds$

$\Rightarrow e^{Q(t)+\mu t}S_2\left(t\right)=e^{Q(0)}S_2\left(0\right)+\int\limits_0^tQ_2e^{Q(s)+\mu s}ds$

$\Rightarrow S_2\left(t\right)=\frac{e^{Q(0)}}{e^{Q(t)+\mu t}}S_2\left(0\right)+\frac{1}{e^{Q(t)+\mu t}}\int\limits_0^tQ_2e^{Q(s)+\mu s}ds$

$\Rightarrow S_2\left(t\right)=S_2\left(0\right)e^{-Q(t)+Q(0)-\mu t}+\int\limits_0^tQ_2e^{(Q(s)-Q(t))+\mu(s-t)}ds$

From this solution we can observe that $S_2(t)$ is positive since $S_2(0)>0, Q_2>0$ and the exponential function always positive.

Thirdly let us consider the differential equation (5.3)

$\frac{dI_1}{dt} = [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\,\frac{S_1}{N} + \sigma\,[I_1 + I_2 + P]\,\frac{S_1}{N} + \sigma\,[I_1 + I_2 + P]\,\frac{S_2}{N}$

$\quad +p_1 I_1 + \ (1-\epsilon)\,\phi I_1 - (k_1 + \theta + \delta_1 + \mu + u_1)\,I_1$

$\Rightarrow \frac{dI_1}{dt} + [K - (\beta_1 + \sigma)\,\frac{S_1}{N} - \sigma\frac{S_2}{N}]I_1 = (\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}$

where $K = (k_1 + \theta + \delta_1 + \mu + u_1) - p_1 - \ (1-\epsilon)\,\phi$

This is a first order linear ordinary differential equation.

We can find the integrating factor

$\mu_1(t) = e^{\int \left[K - (\beta_1+\sigma)\frac{S_1}{N} - \sigma\frac{S_2}{N}\right]dt} = e^{Kt - (\beta_1+\sigma)Q(t) - \sigma M(t)}$ where $Q(t)$ is the anti-derivative of $\frac{S_1(t)}{N(t)}$ and $M(t)$ is the anti-derivative of $\frac{S_2(t)}{N(t)}$

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{Kt-(\beta_1+\sigma)Q(t)-\sigma M(t)}\frac{dI_1}{dt} + [K - (\beta_1+\sigma)\,\frac{S_1}{N} - \sigma\frac{S_2}{N}]e^{Kt-(\beta_1+\sigma)Q(t)-\sigma M(t)}I_1$

$\quad = \left[(\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}\right] e^{Kt-(\beta_1+\sigma)Q(t)-\sigma M(t)}$

$\Rightarrow \left(e^{Kt-(\beta_1+\sigma)Q(t)-\sigma M(t)} I_1(t)\right)'$

$\quad = \left[(\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}\right] e^{Kt-(\beta_1+\sigma)Q(t)-\sigma M(t)}$

Integrating both sides from 0 to $\bar{t}$ will give us

$\int_0^{\bar{t}} \left(e^{Ks-(\beta_1+\sigma)Q(s)-\sigma M(s)} I_1(s)\right)' ds$

$\quad = \int_0^{\bar{t}} \left[(\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}\right] e^{Ks-(\beta_1+\sigma)Q(s)-\sigma M(s)} ds$

$\Rightarrow e^{Ks-(\beta_1+\sigma)Q(s)-\sigma M(s)} I_1(s)\Big|_0^{\bar{t}}$

$\quad = \int_0^{\bar{t}} \left[(\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}\right] e^{Ks-(\beta_1+\sigma)Q(s)-\sigma M(s)} ds$

$\Rightarrow e^{K\bar{t}-(\beta_1+\sigma)Q(\bar{t})-\sigma M(\bar{t})} I_1(\bar{t}) - e^{-(\beta_1+\sigma)Q(0)-M(0)} I_1(0)$

$\quad = \int_0^{\bar{t}} \left[(\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}\right] e^{Ks-(\beta_1+\sigma)Q(s)-\sigma M(s)} ds$

$\Rightarrow e^{K\bar{t}-(\beta_1+\sigma)Q(\bar{t})-\sigma M(\bar{t})} I_1(\bar{t}) = e^{-(\beta_1+\sigma)Q(0)-M(0)} I_1(0)$

$\quad + \int_0^{\bar{t}} \left[(\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}\right] e^{Ks-(\beta_1+\sigma)Q(s)-\sigma M(s)} ds$

$\Rightarrow I_1(\bar{t}) = \frac{e^{-(\beta_1+\sigma)Q(0)-M(0)}}{e^{K\bar{t}-(\beta_1+\sigma)Q(\bar{t})-\sigma M(\bar{t})}} I_1(0)$

$\quad + \frac{1}{e^{K\bar{t}-(\beta_1+\sigma)Q(\bar{t})-\sigma M(\bar{t})}} \int_0^{\bar{t}} \left[(\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}\right] e^{Ks-(\beta_1+\sigma)Q(s)-\sigma M(s)} ds$

$\Rightarrow I_1(\bar{t}) = I_1(0)\, e^{-K\bar{t}+(\beta_1+\sigma)Q(\bar{t})-(\beta_1+\sigma)Q(0)+\sigma M(\bar{t})-M(0)}$

$\quad + e^{-K\bar{t}+(\beta_1+\sigma)Q(\bar{t})+\sigma M(\bar{t})} \int_0^{\bar{t}} \left[(\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}\right] e^{Ks-(\beta_1+\sigma)Q(s)-\sigma M(s)} ds$

$\Rightarrow I_1(\bar{t}) = I_1(0)\, e^{-K\bar{t}+(\beta_1+\sigma)Q(\bar{t})-(\beta_1+\sigma)Q(0)+\sigma M(\bar{t})-M(0)}$

$\quad + \int_0^{\bar{t}} \left[(\beta_2 + \sigma)\frac{I_2 S_1}{N} + (\beta_3 + \sigma)\frac{PS_1}{N} + \sigma\,(I_2 + P)\,\frac{S_2}{N}\right] e^{Ks-K\bar{t}+(\beta_1+\sigma)Q(\bar{t})-(\beta_1+\sigma)Q(s)+\sigma M(\bar{t})-\sigma M(s)} ds$

since $I_1(0) > 0$ and from the definition of $\bar{t}$, we see that $S_1(t) > 0, S_2(t) > 0, I_2(t) > 0, P(t) > 0$ and also the exponential function always positive, then the solution $I_1(\bar{t}) > 0$. Hence, $I_1(\bar{t})$ could not be zero.

Fourthly, let us consider the differential equation (5.4)

$\frac{dI_2}{dt} = p_2 I_2 + \theta I_1 - (k_2 + \delta_2 + \mu + u_2) I_2$

$\Rightarrow \frac{dI_2}{dt} + hI_2 = \theta I_1$ where $h = (k_2 + \delta_2 + \mu + u_2 - p_2)$

This is a first order linear ordinary differential equation.

Now we can find the integrating factor

$\mu_1(t) = e^{\int h dt} = e^{ht}$

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{ht}\frac{dI_2}{dt} + he^{ht} I_2 = e^{ht}\theta I_1$

$\Rightarrow \left(e^{ht} I_2(t)\right)' = e^{ht}\theta I_1$

Integrating both sides from 0 to $\bar{t}$ will give us

$\int_0^{\bar{t}} \left(e^{hs} I_2(s)\right)' ds = \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow \left(e^{hs} I_2(s)\right)\Big|_0^{\bar{t}} = \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow e^{h\bar{t}} I_2(t) - I_2(0) = \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow e^{h\bar{t}} I_2(\bar{t}) = I_2(0) + \int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow I_2(\bar{t}) = \frac{1}{e^{h\bar{t}}} I_2(0) + \frac{1}{e^{h\bar{t}}}\int_0^{\bar{t}} e^{hs}\theta I_1 ds$

$\Rightarrow I_2(\bar{t}) = I_2(0)\, e^{-h\bar{t}} + e^{-h\bar{t}}\int_0^{\bar{t}} e^{hs}\theta I_1 ds$

since $I_2(0) > 0$ and from the definition of $\bar{t}$, we see that $I_1(t) > 0$ and also the exponential function always positive, then the solution $I_2(\bar{t}) > 0$. Hence, $I_2(\bar{t})$ could not be zero.

Fifthly, let us consider the differential equation (5.5).

$\frac{dP}{dt} = u_1 I_1 + u_2 I_2 - (\delta_3 + k_3 + \mu) P$

$\Rightarrow \frac{dP}{dt} + (\delta_3 + k_3 + \mu) P = u_1 I_1 + u_2 I_2$

This is a first order linear ordinary differential equation.

Now we can find the integrating factor

$\mu_1(t) = e^{\int (\delta_3 + k_3 + \mu) dt} = e^{(\delta_3 + k_3 + \mu)t}$

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{(\delta_3 + k_3 + \mu)t}\frac{dP}{dt} + (\delta_3 + k_3 + \mu) e^{(\delta_3 + k_3 + \mu)t} P = e^{(\delta_3 + k_3 + \mu)t}(u_1 I_1 + u_2 I_2)$

$\Rightarrow \left(e^{(\delta_3 + k_3 + \mu)t} P\right)' = e^{(\delta_3 + k_3 + \mu)t}(u_1 I_1 + u_2 I_2)$

Integrating both sides from 0 to $\bar{t}$ will give us

$\int_0^{\bar{t}} \left(e^{(\delta_3 + k_3 + \mu)s} P(s)\right)' ds = \int_0^{\bar{t}} e^{(\delta_3 + k_3 + \mu)s}(u_1 I_1 + u_2 I_2) ds$

$$\Rightarrow \left(e^{(\delta_3+k_3+\mu)s}P(s)\right)\Big|_0^{\bar{t}} = \int_0^{\bar{t}} e^{(\delta_3+k_3+\mu)s}\left(u_1I_1+u_2I_2\right)ds$$

$$\Rightarrow \left(e^{(\delta_3+k_3+\mu)\bar{t}}P(\bar{t})\right) - P(0) = \int_0^{\bar{t}} e^{(\delta_3+k_3+\mu)s}\left(u_1I_1+u_2I_2\right)ds$$

$$\Rightarrow \left(e^{(\delta_3+k_3+\mu)t}P(\bar{t})\right) = P(0) + \int_0^{\bar{t}} e^{(\delta_3+k_3+\mu)s}\left(u_1I_1+u_2I_2\right)ds$$

$$\Rightarrow P(\bar{t}) = \frac{1}{e^{(\delta_3+k_3+\mu)\bar{t}}}P(0) + \frac{1}{e^{(\delta_3+k_3+\mu)\bar{t}}}\int_0^{\bar{t}} e^{(\delta_3+k_3+\mu)s}\left(u_1I_1+u_2I_2\right)ds$$

$$\Rightarrow P(\bar{t}) = P(0)\,e^{-(\delta_3+k_3+\mu)\bar{t}} + e^{-(\delta_3+k_3+\mu)\bar{t}}\int_0^{\bar{t}} e^{(\delta_3+k_3+\mu)s}\left(u_1I_1+u_2I_2\right)ds$$

since $P(0) > 0$ and from the definition of $\bar{t}$, we see that $I_1(t) > 0, I_2(t) > 0$ and also the exponential function always positive, then the solution $P(\bar{t}) > 0$. Hence, $P(\bar{t})$ could not be zero.

Let us consider the differential equation (5.6).

$\frac{dS_p}{dt} = k_1I_1 + k_2I_2 + k_3P - \mu S_p$

$\Rightarrow \frac{dS_p}{dt} + \mu S_p = k_1I_1 + k_2I_2 + k_3P$

This is a first order linear ordinary differential equation.

Now we can find the integrating factor

$\mu_1(t) = e^{\int \mu dt} = e^{\mu t}$

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{\mu t}\frac{dS_p}{dt} + \mu e^{\mu t}S_p = e^{\mu t}\left(k_1I_1 + k_2I_2 + k_3P\right) \Rightarrow \left(e^{\mu t}S_p\right)' = e^{\mu t}\left(k_1I_1 + k_2I_2 + k_3P\right)$

Integrating both sides from 0 to $\bar{t}$ will give us

$$\int_0^{\bar{t}} \left(e^{\mu s}S_p(s)\right)' ds = \int_0^{\bar{t}} e^{\mu s}\left(k_1I_1 + k_2I_2 + k_3P\right)ds$$

$$\Rightarrow \left(e^{\mu s}S_p(s)\right)\Big|_0^{\bar{t}} = \int_0^{\bar{t}} e^{\mu s}\left(k_1I_1 + k_2I_2 + k_3P\right)ds$$

$$\Rightarrow e^{\mu\bar{t}}S_p(\bar{t}) - S_p(0) = \int_0^{\bar{t}} e^{\mu s}\left(k_1I_1 + k_2I_2 + k_3P\right)ds$$

$$\Rightarrow e^{\mu\bar{t}}S_p(\bar{t}) = S_p(0) + \int_0^{\bar{t}} e^{\mu s}\left(k_1I_1 + k_2I_2 + k_3P\right)ds$$

$$\Rightarrow S_p(\bar{t}) = \frac{1}{e^{\mu\bar{t}}}S_p(0) + \frac{1}{e^{\mu\bar{t}}}\int_0^{\bar{t}} e^{\mu s}\left(k_1I_1 + k_2I_2 + k_3P\right)ds$$

$$\Rightarrow S_p(\bar{t}) = S_p(0)\,e^{-\mu\bar{t}} + e^{-\mu\bar{t}}\int_0^{\bar{t}} e^{\mu s}\left(k_1I_1 + k_2I_2 + k_3P\right)ds$$

since $S_p(0) > 0$ and from the definition of $\bar{t}$, we see that $I_1(t) > 0, I_2(t) > 0, P(t) > 0$ and also the exponential function always positive, then the solution $S_p(\bar{t}) > 0$. Hence, $S_p(\bar{t})$ could not be zero.

Finally, let us consider the differential equation (5.7).

$\frac{dA}{dt} = \delta_1I_1 + \delta_2I_2 + \delta_3P - (\alpha+\mu)A$

$\Rightarrow \frac{dA}{dt} + (\alpha+\mu)A = \delta_1I_1 + \delta_2I_2 + \delta_3P$

This is a first order linear ordinary differential equation.

Now we can find the integrating factor

$\mu_1(t) = e^{\int(\alpha+\mu)dt} = e^{(\alpha+\mu)t}$

Multiply all the terms in the differential equation by the integrating factor and do some simplification.

$e^{(\alpha+\mu)t}\frac{dA}{dt} + (\alpha+\mu)e^{(\alpha+\mu)t}A = e^{(\alpha+\mu)t}(\delta_1 I_1 + \delta_2 I_2 + \delta_3 P)$

$\Rightarrow \left(e^{(\alpha+\mu)t}A\right)' = e^{(\alpha+\mu)t}(\delta_1 I_1 + \delta_2 I_2 + \delta_3 P)$

Integrating both sides from 0 to $\bar{t}$ will give us

$\int_0^{\bar{t}} \left(e^{(\alpha+\mu)s}A(s)\right)' ds = \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2 + \delta_3 P)\, ds$

$\Rightarrow \left(e^{(\alpha+\mu)s}A(s)\right)\Big|_0^{\bar{t}} = \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2 + \delta_3 P)\, ds$

$\Rightarrow e^{(\alpha+\mu)\bar{t}}A(\bar{t}) - A(0) = \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2 + \delta_3 P)\, ds$

$\Rightarrow e^{(\alpha+\mu)\bar{t}}A(\bar{t}) = A(0) + \int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2 + \delta_3 P)\, ds$

$\Rightarrow A(\bar{t}) = \frac{1}{e^{(\alpha+\mu)\bar{t}}}A(0) + \frac{1}{e^{(\alpha+\mu)\bar{t}}}\int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2 + \delta_3 P)\, ds$

$\Rightarrow A(\bar{t}) = A(0)e^{-(\alpha+\mu)\bar{t}} + e^{-(\alpha+\mu)\bar{t}}\int_0^{\bar{t}} e^{(\alpha+\mu)s}(\delta_1 I_1 + \delta_2 I_2 + \delta_3 P)\, ds$

since $A(0) > 0$ and from the definition of $\bar{t}$, we see that $I_1(t) > 0, I_2(t) > 0, P(t) > 0$ and also the exponential function always positive, then the solution $A(\bar{t}) > 0$. Hence, $A(\bar{t})$ could not be zero.

Therefore all the state variables at $\bar{t}$ could not be zero, implies that $\bar{t}$ is not finite. Consequently $\bar{t} = +\infty$, so that for all $t \geq 0$, $S_1(t) > 0, S_2(t) > 0, I_1(t) > 0, I_2(t) > 0, P(t) > 0, S_p(t) > 0$, and $A(t) > 0$. By this we have shown that all the solutions of system (5.1) to (5.7) are in $R_+^7$, provided that the initial conditions are positive. □

We now show that all feasible solutions are uniformly bounded in $\Omega$.

**Theorem 5.2** (Boundedness)**.** *The feasible region $\Omega$ of the dynamical system (5.1) to (5.7) is defined as:*

$\Omega = \{(S_1(t), S_2(t), I_1(t), I_2(t), P(t), S_p(t), A(t)) \in R_+^7 : 0 < N(t) \leqslant \frac{Q_1+Q_2}{\mu}\}$ *is bounded.*

*Proof.* We assume that all state variables and parameters are positive.

Here we have $N = S_1 + S_2 + I_1 + I_2 + P + S_p + A$ then

$\frac{dN}{dt} = \frac{dS_1}{dt} + \frac{dS_2}{dt} + \frac{dI_1}{dt} + \frac{dI_2}{dt} + \frac{dP}{dt} + \frac{dS_p}{dt} + \frac{dA}{dt}$

Summing up all the seven equations from systems (5.1) to (5.7) and assuming the inequality $p_1 I_1 + (1-\epsilon)\phi I_1 + p_2 I_2 \leq \alpha A$ we obtain $\frac{dN}{dt} \leq Q_1 + Q_2 - \mu N$

$\Rightarrow \frac{dN}{Q_1+Q_2-\mu N} \leq dt$ integrating both sides

$\int_0^t \frac{dN}{Q_1+Q_2-\mu N} \leq \int_0^t ds$

$\Rightarrow \frac{-1}{\mu}\left[\ln\left(Q_1+Q_2-\mu N\left(t\right)\right)-\ln\left(Q_1+Q_2-\mu N\left(0\right)\right)\right] \leq t$

$\Rightarrow \left[\ln\left(Q_1+Q_2-\mu N\left(t\right)\right)-\ln\left(Q_1+Q_2-\mu N\left(0\right)\right)\right] \geq -\mu t$

$\Rightarrow ln\left[\frac{Q_1+Q_2-\mu N(t)}{Q_1+Q_2-\mu N(0)}\right] \geq -\mu t$

$\Rightarrow \left[\frac{Q_1+Q_2-\mu N(t)}{Q_1+Q_2-\mu N(0)}\right] \geq e^{-\mu t}$

$\Rightarrow Q_1+Q_2-\mu N\left(t\right) \geq e^{-\mu t}\left(Q_1+Q_2-\mu N\left(0\right)\right)$

$\Rightarrow Q_1+Q_2-\mu N\left(t\right) \geq \left(Q_1+Q_2\right)e^{-\mu t}-\mu N\left(0\right)e^{-\mu t}$

$\Rightarrow \mu N\left(t\right) \leq Q_1+Q_2-\left(Q_1+Q_2\right)e^{-\mu t}+\mu N\left(0\right)e^{-\mu t} \leq Q_1+Q_2+\mu N\left(0\right)e^{-\mu t}$

$\Rightarrow N\left(t\right) \leq \frac{Q_1+Q_2}{\mu}+N\left(0\right)e^{-\mu t}$

Thus as $t \longrightarrow \infty$ we have $0 < N(t) \leq \frac{Q_1+Q_2}{\mu}$ which indicates that the total population is bounded. □

## 5.3 Stability Analysis of Disease Free and Endemic Equilibrium Points

In this section we identify the equilibrium points of the model developed in this study and provided as a system of equations from (5.1) to (5.7). We also analyze their stability conditions and present the results. The system exhibits two types of equilibrium points; disease free equilibrium point and endemic equilibrium point.

### 5.3.1 Disease Free Equilibrium Point

The disease free equilibrium point of the model (5.1) to (5.7)is obtained by setting $\frac{dS_1}{dt} = \frac{dS_2}{dt} = \frac{dI_1}{dt} = \frac{dI_2}{dt} = \frac{dP}{dt} = \frac{dS_p}{dt} = \frac{dA}{dt} = 0$. Further at the disease free equilibrium point there are neither

infective people nor AIDS patients. Then

$$\begin{aligned}
\frac{dS_1}{dt} &= Q_1 - [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N} - [I_1 + I_2 + P]\sigma\frac{S_1}{N} - \mu S_1 = 0 \\
\frac{dS_2}{dt} &= Q_2 - [I_1 + I_2 + P]\sigma\frac{S_2}{N} - \mu S_2 = 0 \\
\frac{dI_1}{dt} &= [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N} + [I_1 + I_2 + P]\sigma\frac{S_1}{N} + [I_1 + I_2 + P]\sigma\frac{S_2}{N} + p_1 I_1 \\
&\quad + (1-\epsilon)I_1\phi - (k_1 + \theta + \delta_1 + \mu + u_1)I_1 = 0 \\
\frac{dI_2}{dt} &= p_2 I_2 + I_1\theta - (k_2 + \delta_2 + \mu + u_2)I_2 = 0 \\
\frac{dP}{dt} &= u_1 I_1 + u_2 I_2 - (\delta_3 + k_3 + \mu)P = 0 \\
\frac{dS_p}{dt} &= k_1 I_1 + k_2 I_2 + k_3 P - \mu S_p = 0 \\
\frac{dA}{dt} &= \delta_1 I_1 + \delta_2 I_2 + \delta_3 P - (\alpha + \mu)A = 0
\end{aligned}$$

This system reduced to

$$Q_1 - \mu S_1 = 0$$

$$Q_2 - \mu S_2 = 0$$

Since at disease free we have $I_1 = I_2 = P = A = S_p = 0$

Thus $Q_1 - \mu S_1 = 0$

$\Rightarrow \mu S_1 = Q_1$

$\Rightarrow S_1 = \frac{Q_1}{\mu}$

And $Q_2 - \mu S_2 = 0 \Rightarrow \mu S_2 = Q_2$

$\Rightarrow S_2 = \frac{Q_2}{\mu}$

And hence we obtain the disease free equilibrium point of the dynamical system is $E_0 = (\frac{Q_1}{\mu}, \frac{Q_2}{\mu}, 0, 0, 0, 0, 0)$.

**Basic Reproduction number**

The basic reproduction number is defined as the average number of secondary infections that occur when one infective is introduced into a completely susceptible host population [26]. We can calculate the basic reproduction number, $R_0$, using the next generation approach proposed by van den Driessche and Watmough [85]. According to this approach, in order to compute the basic reproduction number, it is important to distinguish new infections from all other class transitions in the population. The infected classes are $I_1, I_2$, and $P$. We can write system (5.1)-(5.7) as: $\dot{x} = F(x) - V(x), V = V^- - V^+$, where $x = (S_1, S_2, I_1, I_2, P, S_p, A)$. F is the

rate of appearance of new infection in each class, $V^-$ is the rate of transfer into each class by all other means, and $V^+$ is the rate of transfer of the infectious individuals out of each class. Using system of differential equations below

$$\begin{aligned}
\frac{dI_1}{dt} &= [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N} + [I_1 + I_2 + P]\sigma\frac{S_1}{N} + [I_1 + I_2 + P]\sigma\frac{S_2}{N} + p_1 I_1 \\
&\quad + (1-\epsilon)I_1\phi - (k_1 + \theta + \delta_1 + \mu + u_1)I_1 = 0 \\
\frac{dI_2}{dt} &= p_2 I_2 + I_1\theta - (k_2 + \delta_2 + \mu + u_2)I_2 = 0 \\
\frac{dP}{dt} &= u_1 I_1 + u_2 I_2 - (\delta_3 + k_3 + \mu)P = 0 \\
\frac{dA}{dt} &= \delta_1 I_1 + \delta_2 I_2 + \delta_3 P - (\alpha + \mu)A = 0 \\
\frac{dS_p}{dt} &= k_1 I_1 + k_2 I_2 + k_3 P - \mu S_p = 0 \\
\frac{dS_1}{dt} &= Q_1 - [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N} - [I_1 + I_2 + P]\sigma\frac{S_1}{N} - \mu S_1 = 0 \\
\frac{dS_2}{dt} &= Q_2 - [I_1 + I_2 + P]\sigma\frac{S_2}{N} - \mu S_2 = 0
\end{aligned}$$

The associated matrices, $F(x)$ for the new infection terms, and $V(x)$ for the remaining transition terms are respectively given by,

$$F(x) = \begin{bmatrix} [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N} + \sigma[I_1 + I_2 + P]\frac{S_1}{N} + \sigma[I_1 + I_2 + P]\frac{S_2}{N} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V(x) = \begin{bmatrix} (k_1 + \theta + \delta_1 + \mu + u_1)I_1 - p_1 I_1 - (1-\epsilon)\phi I_1 \\ (k_2 + \delta_2 + \mu + u_2)I_2 - p_2 I_2 - \theta I_1 \\ (\delta_3 + k_3 + \mu)P - u_1 I_1 - u_2 I_2 \\ (\alpha + \mu)A - \delta_1 I_1 - \delta_2 I_2 - \delta_3 P \\ \mu S_p - k_1 I_1 - k_2 I_2 - k_3 P \\ \mu S_1 - Q_1 \\ \mu S_2 - Q_2 \end{bmatrix}$$

Evaluating the partial derivatives of $F(x)$ and bearing in mind that system (5.1) − (5.7) has three infected classes, namely $I_1, I_2$ and $P$, we obtain

$$F(x) = \begin{bmatrix} (\beta_1+\sigma)\frac{S_1}{N} + \sigma\frac{S_2}{N} & (\beta_2+\sigma)\frac{S_1}{N} + \sigma\frac{S_2}{N} & (\beta_3+\sigma)\frac{S_1}{N} + \sigma\frac{S_2}{N} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

At disease free equilibrium point we have $S_1 + S_2 \approx N$. Thus

$$F = \begin{bmatrix} \beta_1\frac{S_1}{S_1+S_2} + \sigma & \beta_2\frac{S_1}{S_1+S_2} + \sigma & \beta_3\frac{S_1}{S_1+S_2} + \sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow F = \begin{bmatrix} \beta_1\frac{Q_1}{Q_1+Q_2} + \sigma & \beta_2\frac{Q_1}{Q_1+Q_2} + \sigma & \beta_3\frac{Q_1}{Q_1+Q_2} + \sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarly, the partial derivatives of $V(x)$ with respect to $I_1, I_2$ and $P$ at $E_0$ gives

$$V = \begin{bmatrix} (k_1+\theta+\delta_1+\mu+u_1-p_1-(1-\epsilon)\phi) & 0 & 0 \\ -\theta & (k_2+\delta_2+\mu+u_2-p_2) & 0 \\ -u_1 & -u_2 & (\delta_3+k_3+\mu) \end{bmatrix}$$

To get $V^{-1}$, we use the adjoint matrix method.

$$V^{-1} = \frac{1}{det\,(V)} adj\,(V)$$

Then

$$\det(V) = \begin{vmatrix} \nabla_1 & 0 & 0 \\ -\theta & \nabla_2 & 0 \\ -u_1 & -u_2 & \nabla_3 \end{vmatrix}$$

where $\nabla_1 = (k_1+\theta+\delta_1+\mu+u_1-p_1-\ (1-\epsilon)\,\phi)$, $\nabla_2 = (k_2+\delta_2+\mu+u_2-p_2)$ and $\nabla_3 = (\delta_3+k_3+\mu)$

To get the adjoint matrix of $V$, first we will find the cofactors of each entry of matrix $V$ as follows:

$$C_{11} = (-1)^{1+1}\begin{vmatrix} (k_2+\delta_2+\mu+u_2-p_2) & 0 \\ -u_2 & (\delta_3+k_3+\mu) \end{vmatrix} = \nabla_2\nabla_3$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} -\theta & 0 \\ -u_1 & (\delta_3+k_3+\mu) \end{vmatrix} = \theta\nabla_3$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -\theta & (k_2+\delta_2+\mu+u_2-p_2) \\ -u_1 & -u_2 \end{vmatrix} = \theta u_2 + u_1\nabla_2$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 0 \\ -u_2 & (\delta_3+k_3+\mu) \end{vmatrix} = 0$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} (k_1+\theta+\delta_1+\mu+u_1-p_1-\ (1-\epsilon)\,\phi) & 0 \\ -u_1 & (\delta_3+k_3+\mu) \end{vmatrix} = \nabla_1\nabla_3$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} (k_1+\theta+\delta_1+\mu+u_1-p_1-\ (1-\epsilon)\,\phi) & 0 \\ -u_1 & -u_2 \end{vmatrix} = u_2\nabla_1$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 0 \\ (k_2+\delta_2+\mu+u_2-p_2) & 0 \end{vmatrix} = 0$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} (k_1+\theta+\delta_1+\mu+u_1-p_1-\ (1-\epsilon)\,\phi) & 0 \\ -\theta & 0 \end{vmatrix} = 0$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} (k_1+\theta+\delta_1+\mu+u_1-p_1-\ (1-\epsilon)\,\phi) & 0 \\ -\theta & (k_2+\delta_2+\mu+u_2-p_2) \end{vmatrix} = \nabla_1\nabla_2$$

The matrix formed by the cofactors is

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} \nabla_2\nabla_3 & \theta\nabla_3 & \theta u_2 + u_1\nabla_2 \\ 0 & \nabla_1\nabla_3 & u_2\nabla_1 \\ 0 & 0 & \nabla_1\nabla_2 \end{pmatrix}$$

$$\text{then } adjV = C^T = \begin{pmatrix} \nabla_2\nabla_3 & 0 & 0 \\ \theta\nabla_3 & \nabla_1\nabla_3 & 0 \\ \theta u_2 + u_1\nabla_2 & u_2\nabla_1 & \nabla_1\nabla_2 \end{pmatrix}$$

Therefore

$$V^{-1} = \frac{1}{\det(V)} adj\,(V) = \frac{1}{\nabla_1\nabla_2\nabla_3}\begin{pmatrix} \nabla_2\nabla_3 & 0 & 0 \\ \theta\nabla_3 & \nabla_1\nabla_3 & 0 \\ \theta u_2 + u_1\nabla_2 & u_2\nabla_1 & \nabla_1\nabla_2 \end{pmatrix}$$

$$\Rightarrow V^{-1} = \begin{pmatrix} \frac{1}{\nabla_1} & 0 & 0 \\ \frac{\theta}{\nabla_1\nabla_2} & \frac{1}{\nabla_2} & 0 \\ \frac{\theta u_2 + u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} & \frac{u_2}{\nabla_2\nabla_3} & \frac{1}{\nabla_3} \end{pmatrix}$$

$$FV^{-1} = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\nabla_1} & 0 & 0 \\ \frac{\theta}{\nabla_1\nabla_2} & \frac{1}{\nabla_2} & 0 \\ \frac{\theta u_2 + u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} & \frac{u_2}{\nabla_2\nabla_3} & \frac{1}{\nabla_3} \end{pmatrix}$$

where $A_1 = \beta_1 \frac{Q_1}{Q_1+Q_2} + \sigma, \quad A_2 = \beta_2 \frac{Q_1}{Q_1+Q_2} + \sigma$ *and* $A_3 = \beta_3 \frac{Q_1}{Q_1+Q_2} + \sigma$ then

$$FV^{-1} = \begin{bmatrix} A_1\frac{1}{\nabla_1} + A_2\frac{\theta}{\nabla_1\nabla_2} + A_3\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} & A_2\frac{1}{\nabla_2} + A_3\frac{u_2}{\nabla_2\nabla_3} & A_3\frac{1}{\nabla_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We find the eigenvalues of $FV^{-1}$ by solving the characteristic equation $|FV^{-1} - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} A_1\frac{1}{\nabla_1} + A_2\frac{\theta}{\nabla_1\nabla_2} + A_3\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} - \lambda & A_2\frac{1}{\nabla_2} + A_3\frac{u_2}{\nabla_2\nabla_3} & A_3\frac{1}{\nabla_3} \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)\begin{vmatrix} A_1\frac{1}{\nabla_1} + A_2\frac{\theta}{\nabla_1\nabla_2} + A_3\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} - \lambda & A_2\frac{1}{\nabla_2} + A_3\frac{u_2}{\nabla_2\nabla_3} \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)(-\lambda)\left(A_1\frac{1}{\nabla_1} + A_2\frac{\theta}{\nabla_1\nabla_2} + A_3\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} - \lambda\right) = 0$$

$$\Rightarrow \lambda^2\left(A_1\frac{1}{\nabla_1} + A_2\frac{\theta}{\nabla_1\nabla_2} + A_3\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} - \lambda\right) = 0$$

$\Rightarrow \lambda_{1,2} = 0$ and $\lambda_3 = A_1\frac{1}{\nabla_1} + A_2\frac{\theta}{\nabla_1\nabla_2} + A_3\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}$ thus the spectral radius of $FV^{-1}$ is given by

$$R_0 = \max\left[\lambda_{1,2},\ \lambda_3\right] = \lambda_3$$

Therefore the basic reproduction number of the model is

$$R_0 = A_1\frac{1}{\nabla_1} + A_2\frac{\theta}{\nabla_1\nabla_2} + A_3\frac{\theta u_2 + u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}$$

From this we can observe that the contribution to the reproduction number by unaware infective $I_1$ is $R_0^{I_1} = A_1\frac{1}{\nabla_1}$ , the contribution to the reproduction number by aware infective $I_2$ is $R_0^{I_2} = A_2\frac{\theta}{\nabla_1\nabla_2}$, and the contribution to the reproduction number by pre-AIDS individuals $P$ is $R_0^P = A_3\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}$.

We now investigate the local stability of the disease free equilibrium point.

**Local stability of the disease free equilibrium point** $E_0$

**Theorem 5.3.** *The disease free equilibrium point $E_0$ of the system of ordinary differential equations (5.1)−(5.7) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

*Proof.* Initially at $t = 0,\ S_1(0) > 0, S_2(0) > 0, I_1(0) \geq 0, I_2(0) \geq 0, P(0) \geq 0, S_p(0) \geq 0$ this means initially there is no AIDS patient. Hence, we only consider the subsystem of six equations (5.1 to 5.6). The Jacobian matrix associated with the subsystem equations at the

disease free equilibrium point $E_0 = (\frac{Q_1}{\mu}, \frac{Q_2}{\mu}, 0, 0, 0, 0, 0)$ is given by:

$$J(E_0) = \begin{bmatrix} -\mu & 0 & -(\beta_1+\sigma)\frac{Q_1}{Q_1+Q_2} & -(\beta_2+\sigma)\frac{Q_1}{Q_1+Q_2} & -(\beta_3+\sigma)\frac{Q_1}{Q_1+Q_2} & 0 \\ 0 & -\mu & -\sigma\frac{Q_2}{Q_1+Q_2} & -\sigma\frac{Q_2}{Q_1+Q_2} & -\sigma\frac{Q_2}{Q_1+Q_2} & 0 \\ 0 & 0 & A_1-\nabla_1 & A_2 & A_3 & 0 \\ 0 & 0 & \theta & -\nabla_2 & 0 & 0 \\ 0 & 0 & u_1 & u_2 & -\nabla_3 & 0 \\ 0 & 0 & k_1 & k_2 & k_3 & -\mu \end{bmatrix}$$

The characteristic equation $|J(E_0) - \lambda I| = 0$

$$\begin{vmatrix} -\mu-\lambda & 0 & -(\beta_1+\sigma)\frac{Q_1}{Q_1+Q_2} & -(\beta_2+\sigma)\frac{Q_1}{Q_1+Q_2} & -(\beta_3+\sigma)\frac{Q_1}{Q_1+Q_2} & 0 \\ 0 & -\mu-\lambda & -\sigma\frac{Q_2}{Q_1+Q_2} & -\sigma\frac{Q_2}{Q_1+Q_2} & -\sigma\frac{Q_2}{Q_1+Q_2} & 0 \\ 0 & 0 & A_1-\nabla_1-\lambda & A_2 & A_3 & 0 \\ 0 & 0 & \theta & -\nabla_2-\lambda & 0 & 0 \\ 0 & 0 & u_1 & u_2 & -\nabla_3-\lambda & 0 \\ 0 & 0 & k_1 & k_2 & k_3 & -\mu-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\mu+\lambda)^3 \begin{vmatrix} A_1-\nabla_1-\lambda & A_2 & A_3 \\ \theta & -\nabla_2-\lambda & 0 \\ u_1 & u_2 & -\nabla_3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\mu+\lambda)^3 \left[ -\theta \begin{vmatrix} A_2 & A_3 \\ u_2 & -\nabla_3-\lambda \end{vmatrix} - (\nabla_2+\lambda) \begin{vmatrix} A_1-\nabla_1-\lambda & A_3 \\ u_1 & -\nabla_3-\lambda \end{vmatrix} \right] = 0$$

$$\Rightarrow (\mu+\lambda)^3 \left[\theta\left(A_2\left(\nabla_3+\lambda\right)+u_2A_3\right)+\left(\nabla_2+\lambda\right)\left(\left(A_1-\nabla_1-\lambda\right)\left(\nabla_3+\lambda\right)+u_1A_3\right)\right] = 0$$

$$\Rightarrow \lambda = -\mu \text{ or } \left[\theta\left(A_2\left(\nabla_3+\lambda\right)+u_2A_3\right)+\left(\nabla_2+\lambda\right)\left(\left(A_1-\nabla_1-\lambda\right)\left(\nabla_3+\lambda\right)+u_1A_3\right)\right] = 0$$

$$\Rightarrow \theta A_2\nabla_3 + \theta A_2\lambda + \theta u_2A_3 + \left(\nabla_2+\lambda\right)\left(\nabla_3+\lambda\right)\left(A_1-\nabla_1-\lambda\right) + \left(\nabla_2+\lambda\right)u_1A_3 = 0$$

$$\Rightarrow \theta A_2\nabla_3 + \theta A_2\lambda + \theta u_2A_3 + \left(\nabla_2\nabla_3 + \left(\nabla_2+\nabla_3\right)\lambda + \lambda^2\right)\left(A_1-\nabla_1-\lambda\right) + \nabla_2u_1A_3 + \lambda u_1A_3 = 0$$

$$\Rightarrow \theta A_2\nabla_3 + \theta A_2\lambda + \theta u_2A_3 + A_1\nabla_2\nabla_3 + A_1\left(\nabla_2+\nabla_3\right)\lambda + A_1\lambda^2 - \nabla_1\nabla_2\nabla_3 - \nabla_1\left(\nabla_2+\nabla_2\right)\lambda$$
$$-\nabla_1\lambda^2 - \lambda\nabla_2\nabla_3 - \left(\nabla_2+\nabla_3\right)\lambda^2 - \lambda^3 + \nabla_2u_1A_3 + \lambda u_1A_3 = 0$$

$$\Rightarrow \lambda^3 + \left(\nabla_1+\nabla_2+\nabla_3-A_1\right)\lambda^2 + \left(\nabla_1\nabla_2+\nabla_1\nabla_3+\nabla_2\nabla_3-\theta A_2-u_1A_3-A_1\nabla_2-A_1\nabla_3\right)\lambda$$
$$+\left(\nabla_1\nabla_2\nabla_3 - A_1\nabla_2\nabla_3 - \nabla_2u_1A_3 - \theta u_2A_3 - \theta A_2\nabla_3\right) = 0$$

Let $B_1 = (\nabla_1+\nabla_2+\nabla_3-A_1)$

$B_2 = (\nabla_1\nabla_2+\nabla_1\nabla_3+\nabla_2\nabla_3-\theta A_2-u_1A_3-A_1\nabla_2-A_1\nabla_3)$

$B_3 = (\nabla_1\nabla_2\nabla_3 - A_1\nabla_2\nabla_3 - \nabla_2u_1A_3 - \theta u_2A_3 - \theta A_2\nabla_3)$

Then the characteristics polynomial equation becomes

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$$

If we show that $B_1, B_2$ and $B_3$ are positives, then the necessary condition of Routh wuritze criteria satisfied.

Now

$$R_0 = A_1\frac{1}{\nabla_1} + A_2\frac{\theta}{\nabla_1\nabla_2} + A_3\frac{\theta u_2 + u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} = \frac{A_1\nabla_2\nabla_3 + A_2\theta\nabla_3 + A_3\theta u_2 + A_3u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}$$

For $R_0 < 1$ we have

$\frac{A_1\nabla_2\nabla_3+A_2\theta\nabla_3+A_3\theta u_2+A_3u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} < 1$

$\Rightarrow \frac{A_1\nabla_2\nabla_3}{\nabla_1\nabla_2\nabla_3} < 1 \Rightarrow \frac{A_1}{\nabla_1} < 1$

$\Rightarrow A_1 < \nabla_1 \Rightarrow \nabla_1 - A_1 > 0$

$\Rightarrow B_1 > 0$ Since $B_1 = \nabla_1 + \nabla_2 + \nabla_3 - A_1 = \nabla_2 + \nabla_3 + \nabla_1 - A_1$ where $\nabla_2 > 0$ and $\nabla_3 > 0$.

$B_2 = (\nabla_1\nabla_2 + \nabla_1\nabla_3 + \nabla_2\nabla_3 - \theta A_2 - u_1A_3 - A_1\nabla_2 - A_1\nabla_3)$

$= \nabla_2(\nabla_1 - A_1) + \nabla_3(\nabla_1 - A_1) + \nabla_2\nabla_3 - (\theta A_2 + u_1A_3)$

Thus $B_2 > 0$ when $\nabla_2\nabla_3 > (\theta A_2 + u_1A_3)$

$B_3 = (\nabla_1\nabla_2\nabla_3 - A_1\nabla_2\nabla_3 - \nabla_2u_1A_3 - \theta u_2A_3 - \theta A_2\nabla_3)$

$= \nabla_1\nabla_2\nabla_3\left(1 - \frac{A_1}{\nabla_1} - \frac{u_1A_3}{\nabla_1\nabla_3} - \frac{\theta u_2A_3}{\nabla_1\nabla_2\nabla_3} - \frac{\theta A_2}{\nabla_1\nabla_2}\right)$

$= \nabla_1\nabla_2\nabla_3\left(1 - \left(\frac{A_1}{\nabla_1} + \frac{u_1A_3}{\nabla_1\nabla_3} + \frac{\theta u_2A_3}{\nabla_1\nabla_2\nabla_3} + \frac{\theta A_2}{\nabla_1\nabla_2}\right)\right)$

$= \nabla_1\nabla_2\nabla_3(1 - R_0)$

If $R_0 < 1$, then $B_3 > 0$.

Thus all coefficients of the characteristic polynomial are positives for $R_0 < 1$.

Now consider the following Routh array to determine the sufficient condition.

To see the sign of eigenvalues we use Routh – Hurwitz criteria.

Consider the following Routh – Hurwitz array

| | | |
|---|---|---|
| $\lambda^3$ | 1 | $B_2$ |
| $\lambda^2$ | $B_1$ | $B_3$ |
| $\lambda^1$ | B | 0 |
| $\lambda^0$ | C | |

where $B = \frac{-1}{B_1}\begin{vmatrix} 1 & B_2 \\ B_1 & B_3 \end{vmatrix}$ and $C = \frac{-1}{B}\begin{vmatrix} B_1 & B_3 \\ B & 0 \end{vmatrix}$

Now

$$B = \frac{-1}{B_1}\begin{vmatrix} 1 & B_2 \\ B_1 & B_3 \end{vmatrix} = \frac{-1}{B_1}(B_3 - B_1B_2)$$

$$= \frac{1}{B_1}(B_1B_2 - B_3)$$

For $B_1B_2 - B_3 > 0 \Rightarrow B_1B_2 > B_3$ we have $B > 0$

$$C = \frac{-1}{B}\begin{vmatrix} B_1 & B_3 \\ B & 0 \end{vmatrix} = \frac{-1}{B}(0 - BB_3)$$
$$= \frac{BB_3}{B} = B_3 > 0$$
$$\Rightarrow C > 0$$

Since all elements of the first column of the array have the same sign then by Routh – Hurwitz criteria all roots of the characteristic equation have negative real part, thus the disease free equilibrium point is locally asymptotically stable. □

**Global stability of disease-free equilibrium point**

**Theorem 5.4.** *The disease free equilibrium point $E_0$ is globally asymptotically stable if $R_0 < 1$.*

*Proof.* Let us construct a Lyapunov function

$$V = \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 P + \alpha_4 S_p + \alpha_5 A$$

where $\alpha_i, i = 1, 2, 3, 4, 5$ are positive constants to be determined. The time derivative of V is given by

$$\begin{aligned}
\tfrac{dV}{dt} &= \alpha_1 \tfrac{dI_1}{dt} + \alpha_2 \tfrac{dI_2}{dt} + \alpha_3 \tfrac{dP}{dt} + \alpha_4 \tfrac{dS_p}{dt} + \alpha_5 \tfrac{dA}{dt} \\
&= \alpha_1 \left( [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P] \tfrac{S_1}{N} + \sigma [I_1 + I_2 + P] \tfrac{S_1}{N} + \sigma [I_1 + I_2 + P] \tfrac{S_2}{N} - \nabla_1 I_1 \right) \\
&\quad + \alpha_2 (p_2 I_2 + \theta I_1 - (k_2 + \delta_2 + \mu + u_2) I_2) + \alpha_3 (u_1 I_1 + u_2 I_2 - (\delta_3 + k_3 + \mu) P) \\
&\quad + \alpha_4 (k_1 I_1 + k_2 I_2 + k_3 P - \mu S_p) + \alpha_5 (\delta_1 I_1 + \delta_2 I_2 + \delta_3 P - (\alpha + \mu) A) \\
&= \alpha_1 \left( (\beta_1 + \sigma) \tfrac{S_1}{N} + \sigma \tfrac{S_2}{N} - \nabla_1 \right) I_1 + \alpha_1 \left( (\beta_2 + \sigma) \tfrac{S_1}{N} + \sigma \tfrac{S_2}{N} \right) I_2 + \alpha_1 \left( (\beta_3 + \sigma) \tfrac{S_1}{N} + \sigma \tfrac{S_2}{N} \right) P \\
&\quad + \alpha_2 (p_2 - (k_2 + \delta_2 + \mu + u_2)) I_2 + \alpha_2 \theta I_1 + \alpha_3 u_1 I_1 + \alpha_3 u_2 I_2 - \alpha_3 (\delta_3 + k_3 + \mu) P \\
&\quad + \alpha_4 k_1 I_1 + \alpha_4 k_2 I_2 - \alpha_4 \mu S_p + \alpha_5 \delta_1 I_1 + \alpha_5 \delta_2 I_2 + \alpha_5 \delta_3 P - \alpha_5 (\alpha + \mu) A \\
&= \alpha_1 \left( (\beta_1 + \sigma) \tfrac{S_1}{N} + \sigma \tfrac{S_2}{N} - \nabla_1 \right) I_1 + \alpha_2 \theta I_1 + \alpha_3 u_1 I_1 + \alpha_4 k_1 I_1 + \alpha_5 \delta_1 I_1 \\
&\quad + \alpha_1 \left( (\beta_2 + \sigma) \tfrac{S_1}{N} + \sigma \tfrac{S_2}{N} \right) I_2 + \alpha_2 (p_2 - (k_2 + \delta_2 + \mu + u_2)) I_2 + \alpha_3 u_2 I_2 + \alpha_4 k_2 I_2 \\
&\quad + \alpha_5 \delta_2 I_2 + \alpha_1 \left( (\beta_3 + \sigma) \tfrac{S_1}{N} + \sigma \tfrac{S_2}{N} \right) P - \alpha_3 (\delta_3 + k_3 + \mu) P + \alpha_5 \delta_3 P - \alpha_4 \mu S_p - \alpha_5 (\alpha + \mu) A \\
&= \left[ \alpha_1 \left( (\beta_1 + \sigma) \tfrac{S_1}{N} + \sigma \tfrac{S_2}{N} - \nabla_1 \right) + \alpha_2 \theta + \alpha_3 u_1 + \alpha_4 k_1 + \alpha_5 \delta_1 \right] I_1 \\
&\quad + \left[ \alpha_1 \left( (\beta_2 + \sigma) \tfrac{S_1}{N} + \sigma \tfrac{S_2}{N} \right) + \alpha_2 (p_2 - (k_2 + \delta_2 + \mu + u_2)) + \alpha_3 u_2 + \alpha_4 k_2 + \alpha_5 \delta_2 \right] I_2 \\
&\quad + \left[ \alpha_1 \left( (\beta_3 + \sigma) \tfrac{S_1}{N} + \sigma \tfrac{S_2}{N} \right) - \alpha_3 (\delta_3 + k_3 + \mu) + \alpha_5 \delta_3 \right] P - \alpha_4 \mu S_p - \alpha_5 (\alpha + \mu) A \\
&\leq [\alpha_1 (A_1 - \nabla_1) + \alpha_2 \theta + \alpha_3 u_1 + \alpha_4 k_1 + \alpha_5 \delta_1] I_1
\end{aligned}$$

$$+\left[\alpha_1 A_2+\alpha_2\left(p_2-(k_2+\delta_2+\mu+u_2)\right)+\alpha_3 u_2+\alpha_4 k_2+\alpha_5\delta_2\right] I_2$$
$$+\left[\alpha_1 A_3-\alpha_3\left(\delta_3+k_3+\mu\right)+\alpha_5\delta_3\right] P-\alpha_4\mu S_p-\alpha_5\left(\alpha+\mu\right) A$$

Take the coefficients of $I_2, P, S_p$ and $A$ are equal to zero. Then we get

$$-\alpha_5\left(\alpha+\mu\right) A=0 \Rightarrow \alpha_5=0$$

$$-\alpha_4\mu S_p=0 \Rightarrow \alpha_4=0$$

$$\alpha_1 A_3-\alpha_3\left(\delta_3+k_3+\mu\right)+\alpha_5\delta_3=0 \Rightarrow \alpha_1 A_3-\alpha_3\left(\delta_3+k_3+\mu\right)=0$$

$$\Rightarrow \alpha_1 A_3=\alpha_3\left(\delta_3+k_3+\mu\right)$$
$$\Rightarrow \alpha_3=\frac{\alpha_1 A_3}{\left(\delta_3+k_3+\mu\right)}=\frac{\alpha_1 A_3}{\nabla_3}$$
$$\alpha_1 A_2+\alpha_2\left(p_2-(k_2+\delta_2+\mu+u_2)\right)+\alpha_3 u_2+\alpha_4 k_2+\alpha_5\delta_2=0$$

$$\Rightarrow \alpha_1 A_2+\alpha_2\left(p_2-(k_2+\delta_2+\mu+u_2)\right)+\alpha_3 u_2=0$$
$$\Rightarrow \alpha_1 A_2+\alpha_2\left(p_2-(k_2+\delta_2+\mu+u_2)\right)+\frac{\alpha_1 A_3}{\nabla_3} u_2=0$$
$$\Rightarrow \frac{\alpha_1 A_2\nabla_3+\alpha_1 A_3 u_2}{\nabla_3}-\alpha_2\nabla_2=0$$
$$\Rightarrow \frac{\alpha_1 A_2\nabla_3+\alpha_1 A_3 u_2}{\nabla_3}=\alpha_2\nabla_2$$
$$\Rightarrow \alpha_2=\frac{\alpha_1\left[A_2\nabla_3+A_3 u_2\right]}{\nabla_2\nabla_3}$$

Then

$$\frac{dV}{dt} \leq [\alpha_1(A_1-\nabla_1)+\alpha_2\theta+\alpha_3 u_1+\alpha_4 k_1+\alpha_5\delta_1] I_1$$
$$\Rightarrow \frac{dV}{dt} \leq \left[\alpha_1\left(A_1-\nabla_1\right)+\frac{\alpha_1\left[A_2\nabla_3+A_3 u_2\right]}{\nabla_2\nabla_3}\theta+\frac{\alpha_1 A_3}{\nabla_3} u_1\right] I_1$$
$$\Rightarrow \frac{dV}{dt} \leq \left[\alpha_1\left(A_1-\nabla_1\right)+\frac{\alpha_1\left[A_2\nabla_3+A_3 u_2\right]\theta+\alpha_1 A_3 u_1\nabla_2}{\nabla_2\nabla_3}\right] I_1$$
$$\Rightarrow \frac{dV}{dt} \leq \left[\alpha_1 A_1+\frac{\alpha_1\left[A_2\nabla_3+A_3 u_2\right]\theta+\alpha_1 A_3 u_1\nabla_2}{\nabla_2\nabla_3}-\alpha_1\nabla_1\right] I_1$$
$$\Rightarrow \frac{dV}{dt} \leq \left[\alpha_1\nabla_1\left(\frac{A_1\nabla_2\nabla_3+\left[A_2\nabla_3+A_3 u_2\right]\theta+A_3 u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}-1\right)\right] I_1$$
$$\Rightarrow \frac{dV}{dt} \leq [\alpha_1(A_1-\nabla_1)+\alpha_2\theta+\alpha_3 u_1] I_1$$
$$\Rightarrow \frac{dV}{dt} \leq [\alpha_1\nabla_1\frac{(A_1\nabla_2\nabla_3+A_2\theta\nabla_3+A_3[u_1\nabla_2+\theta u_2])}{\nabla_1\nabla_2\nabla_3}-1] I_1$$
$$\Rightarrow \frac{dV}{dt} \leq [\alpha_1\nabla_1(R_0-1)] I_1$$

We note that $\frac{dV}{dt} \leq 0$ if $R_0<1$. Furthermore, $\frac{dV}{dt}=0$ if and only if $I_1=I_2=P=S_p=A=0$. Therefore, the largest compact invariant set in $\{(S_1, S_2, I_1, I_2, P, S_p, A) \in \Omega: \frac{dV}{dt}=0\}$, where $R_0<1$ is the singleton $\{E_0\}$. LaSalle's (1976) invariance principle then implies that $E_0$ is globally stable in $\Omega$ if $R_0<1$ otherwise it is unstable. □

### 5.3.2 Endemic Equilibrium Point

Similarly here we also consider the system equations (5.1) − (5.7). At the endemic equilibrium point $E^*$ the disease persists or exists. It is given by $E^* = (S_1^*, S_2^*, I_1^*, I_2^*, P^*, S_p^*, A^*)$. We set each right hand side in system equations to zero and express $I_2^*, P^*, S_p^*, A^*$ in terms of $I_1^*$.

$$\begin{aligned}
\frac{dS_1}{dt} &= Q_1 - [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N} - [I_1 + I_2 + P]\sigma\frac{S_1}{N} - \mu S_1 = 0 \\
\frac{dS_2}{dt} &= Q_2 - [I_1 + I_2 + P]\sigma\frac{S_2}{N} - \mu S_2 = 0 \\
\frac{dI_1}{dt} &= [\beta_1 I_1 + \beta_2 I_2 + \beta_3 P]\frac{S_1}{N} + [I_1 + I_2 + P]\sigma\frac{S_1}{N} + [I_1 + I_2 + P]\sigma\frac{S_2}{N} + p_1 I_1 \\
&\quad + (1-\epsilon)I_1\phi - (k_1 + \theta + \delta_1 + \mu + u_1)I_1 = 0 \\
\frac{dI_2}{dt} &= p_2 I_2 + I_1\theta - (k_2 + \delta_2 + \mu + u_2)I_2 = 0 \\
\frac{dP}{dt} &= u_1 I_1 + u_2 I_2 - (\delta_3 + k_3 + \mu)P = 0 \\
\frac{dS_p}{dt} &= k_1 I_1 + k_2 I_2 + k_3 P - \mu S_p = 0 \\
\frac{dA}{dt} &= \delta_1 I_1 + \delta_2 I_2 + \delta_3 P - (\alpha + \mu)A = 0
\end{aligned}$$

Take the following force of infections at endemic equilibrium point

$$\Lambda_1^* = \frac{(\beta_1 + \sigma)\, {I_1}^* + (\beta_2 + \sigma)\, {I_2}^* + (\beta_3 + \sigma)\, P^*}{N^*} \tag{5.9}$$

$$\Lambda_2^* = \frac{\sigma\,({I_1}^* + {I_2}^* + P^*)}{N^*} \tag{5.10}$$

From equation (5.4) we have

$${I_2}^* = \frac{\theta {I_1}^*}{\nabla_2} = \omega_1 {I_1}^* \tag{5.11}$$

From equation (5.5) we have

$$P^* = \frac{u_1 {I_1}^* + u_2 {I_2}^*}{\nabla_3} = \frac{u_1 {I_1}^* + u_2 \frac{\theta {I_1}^*}{\nabla_2}}{\nabla_3} = \left(\frac{u_1\nabla_2 + u_2\theta}{\nabla_2\nabla_3}\right){I_1}^* = \omega_2 {I_1}^* \tag{5.12}$$

From equation (5.6) we have

$$S_p^* = \frac{k_1 {I_1}^* + k_2 {I_2}^* + k_3 P^*}{\mu} = \frac{k_1 {I_1}^* + k_2\omega_1 {I_1}^* + k_3\omega_2 {I_1}^*}{\mu} = \omega_3 {I_1}^* \tag{5.13}$$

From equation (5.7) we have

$$A^* = \frac{\delta_1 {I_1}^* + \delta_2 {I_2}^* + \delta_3 P^*}{(\alpha + \mu)} = \frac{\delta_1 {I_1}^* + \delta_2\omega_1 {I_1}^* + \delta_3\omega_2 {I_1}^*}{(\alpha + \mu)} = \omega_4 {I_1}^* \tag{5.14}$$

where $\omega_1 = \frac{\theta}{\nabla_2}$, $\omega_2 = \left(\frac{u_1\nabla_2 + u_2\theta}{\nabla_2\nabla_3}\right)$, $\omega_3 = \left(\frac{k_1 + k_2\omega_1 + k_3\omega_2}{\mu}\right)$ and $\omega_4 = \left(\frac{\delta_1 + \delta_2\omega_1 + \delta_3\omega_2}{\alpha+\mu}\right)$

From equation (5.9) we have

$$\begin{aligned}\Lambda_1^* &= \frac{(\beta_1+\sigma)\,{I_1}^* + (\beta_2+\sigma)\,{I_2}^* + (\beta_3+\sigma)\,P^*}{N^*} \\ &= \frac{(\beta_1+\sigma)\,{I_1}^* + (\beta_2+\sigma)\,\omega_1{I_1}^* + (\beta_3+\sigma)\,\omega_2{I_1}^*}{N^*} = \frac{\Phi_1{I_1}^*}{N^*}\end{aligned} \tag{5.15}$$

where $\Phi_1 = (\beta_1+\sigma) + (\beta_2+\sigma)\,\omega_1 + (\beta_3+\sigma)\,\omega_2$

From equation (5.10) we have

$$\Lambda_2^* = \frac{\sigma\,({I_1}^* + {I_2}^* + P^*)}{N^*} = \frac{\sigma\,({I_1}^* + \omega_1{I_1}^* + \omega_2{I_1}^*)}{N^*} = \frac{\Phi_2{I_1}^*}{N^*} \tag{5.16}$$

where $\Phi_2 = \sigma\,(1 + \omega_1 + \omega_2)$

From equation (5.3) we have

$$\Lambda_1^*S_1^* + \Lambda_2^*S_2^* - \nabla_1{I_1}^* = 0 \tag{5.17}$$

From equation (5.1) we have

$$Q_1 - \Lambda_1^*S_1^* - \mu S_1^* = 0 \Rightarrow S_1^* = \frac{Q_1}{\Lambda_1^* + \mu} \tag{5.18}$$

From equation (5.2) we have

$$Q_2 - \Lambda_2^*S_2^* - \mu S_2^* = 0 \Rightarrow S_2^* = \frac{Q_2}{\Lambda_2^* + \mu} \tag{5.19}$$

Substituting equations (5.15), (5.16), (5.18) and (5.19) in to equation (5.17) we get

$\Lambda_1^*S_1^* + \Lambda_2^*S_2^* - \nabla_1{I_1}^* = 0$

$\Rightarrow \Lambda_1^*\left(\frac{Q_1}{\Lambda_1^*+\mu}\right) + \Lambda_2^*\left(\frac{Q_2}{\Lambda_2^*+\mu}\right) - \nabla_1{I_1}^* = 0$

$\Rightarrow \Lambda_1^*Q_1\,(\Lambda_2^*+\mu) + \Lambda_2^*Q_2\,(\Lambda_1^*+\mu) - \nabla_1\,(\Lambda_1^*+\mu)\,(\Lambda_2^*+\mu)\,{I_1}^* = 0$

$\Rightarrow \Lambda_1^*\Lambda_2^*Q_1 + \Lambda_1^*\mu Q_1 + \Lambda_1^*\Lambda_2^*Q_2 + \Lambda_2^*\mu Q_2 - \nabla_1\,(\Lambda_1^*\Lambda_2^* + \Lambda_1^*\mu + \Lambda_2^*\mu + \mu^2)\,{I_1}^* = 0$

$\Rightarrow \Lambda_1^*\Lambda_2^*\,(Q_1+Q_2) + \Lambda_1^*\mu Q_1 + \Lambda_2^*\mu Q_2 - \nabla_1\Lambda_1^*\Lambda_2^*{I_1}^* - \nabla_1\Lambda_1^*\mu{I_1}^* - \nabla_1\Lambda_2^*\mu{I_1}^* - \nabla_1\mu^2{I_1}^* = 0$

$\Rightarrow \Lambda_1^*\Lambda_2^*\,(Q_1+Q_2) - \Lambda_1^*\Lambda_2^*\nabla_1{I_1}^* + \Lambda_1^*\mu Q_1 - \Lambda_1^*\nabla_1\mu{I_1}^* + \Lambda_2^*\mu Q_2 - \Lambda_2^*\nabla_1\mu{I_1}^* - \nabla_1\mu^2{I_1}^* = 0$

$\Rightarrow \Lambda_1^*\Lambda_2^*\,(Q_1+Q_2-\nabla_1{I_1}^*) + \Lambda_1^*\,(\mu Q_1 - \nabla_1\mu{I_1}^*) + \Lambda_2^*\,(\mu Q_2 - \nabla_1\mu{I_1}^*) - \nabla_1\mu^2{I_1}^* = 0$

$\Rightarrow \frac{\Phi_1\Phi_2{I_1}^{*2}}{N^{*2}}\,(Q_1+Q_2-\nabla_1{I_1}^*) + \frac{\Phi_1{I_1}^*}{N^*}\,(\mu Q_1 - \nabla_1\mu{I_1}^*) + \frac{\Phi_2{I_1}^*}{N^*}\,(\mu Q_2 - \nabla_1\mu{I_1}^*) - \nabla_1\mu^2{I_1}^* = 0$

$\Rightarrow \Phi_1\Phi_2{I_1}^{*2}\,(Q_1+Q_2-\nabla_1{I_1}^*) + \Phi_1{I_1}^*N^*\,(\mu Q_1 - \nabla_1\mu{I_1}^*) + \Phi_2{I_1}^*N^*\,(\mu Q_2 - \nabla_1\mu{I_1}^*)$
$-N^{*2}\nabla_1\mu^2{I_1}^* = 0$

$\Rightarrow {I_1}^*[\Phi_1\Phi_2{I_1}^*\,(Q_1+Q_2-\nabla_1{I_1}^*) + \Phi_1N^*\,(\mu Q_1 - \nabla_1\mu{I_1}^*) + \Phi_2N^*\,(\mu Q_2 - \nabla_1\mu{I_1}^*)$
$-N^{*2}\nabla_1\mu^2] = 0$

$\Rightarrow {I_1}^* = 0$ which is DFE or

$\left[\Phi_1\Phi_2{I_1}^*\,(Q_1+Q_2-\nabla_1{I_1}^*) + \Phi_1N^*\,(\mu Q_1 - \nabla_1\mu{I_1}^*) + \Phi_2N^*\,(\mu Q_2 - \nabla_1\mu{I_1}^*) - N^{*2}\nabla_1\mu^2\right] = 0$

$$\Rightarrow \Phi_1\Phi_2{I_1}^*\left(Q_1+Q_2-\nabla_1{I_1}^*\right)+\Phi_1\tfrac{(Q_1+Q_2)}{\mu}\left(\mu Q_1-\nabla_1\mu{I_1}^*\right)+\Phi_2\tfrac{(Q_1+Q_2)}{\mu}\left(\mu Q_2-\nabla_1\mu{I_1}^*\right)$$
$$-\tfrac{(Q_1+Q_2)}{\mu^2}{}^2\nabla_1\mu^2=0$$
$$\Rightarrow \Phi_1\Phi_2{I_1}^*\left(Q_1+Q_2-\nabla_1{I_1}^*\right)+\Phi_1\left(Q_1+Q_2\right)Q_1-\Phi_1\left(Q_1+Q_2\right)\nabla_1{I_1}^*+\Phi_2\left(Q_1+Q_2\right)Q_2$$
$$-\Phi_2\left(Q_1+Q_2\right)\nabla_1{I_1}^*-\left(Q_1+Q_2\right)^2\nabla_1=0$$
$$\Rightarrow \Phi_1\Phi_2\left(Q_1+Q_2\right){I_1}^*-\Phi_1\Phi_2\nabla_1{I_1}^{*2}+\Phi_1\left(Q_1+Q_2\right)Q_1-\Phi_1\left(Q_1+Q_2\right)\nabla_1{I_1}^*+\Phi_2\left(Q_1+Q_2\right)Q_2$$
$$-\Phi_2\left(Q_1+Q_2\right)\nabla_1{I_1}^*-\left(Q_1+Q_2\right)^2\nabla_1=0$$
$$\Rightarrow \Phi_1\Phi_2\nabla_1{I_1}^{*2}-\Phi_1\Phi_2\left(Q_1+Q_2\right){I_1}^*-\Phi_1\left(Q_1+Q_2\right)Q_1+\Phi_1\left(Q_1+Q_2\right)\nabla_1{I_1}^*-\Phi_2\left(Q_1+Q_2\right)Q_2$$
$$+\Phi_2\left(Q_1+Q_2\right)\nabla_1{I_1}^*+\left(Q_1+Q_2\right)^2\nabla_1=0$$
$$\Rightarrow \Phi_1\Phi_2\nabla_1{I_1}^{*2}+\left(\Phi_1\left(Q_1+Q_2\right)\nabla_1+\Phi_2\left(Q_1+Q_2\right)\nabla_1-\Phi_1\Phi_2\left(Q_1+Q_2\right)\right){I_1}^*$$
$$+\left(Q_1+Q_2\right)\left(-\Phi_1Q_1-\Phi_2Q_2+\left(Q_1+Q_2\right)\nabla_1\right)=0$$
$$\Rightarrow \mathrm{A}{I_1}^{*2}+B{I_1}^*+C=0$$

where $A=\Phi_1\Phi_2\nabla_1$

$$B=\left(Q_1+Q_2\right)\left(\Phi_1\nabla_1+\Phi_2\nabla_1-\Phi_1\Phi_2\right)$$
$$C=\left(Q_1+Q_2\right)\left(\nabla_1\left(Q_1+Q_2\right)-\left(\Phi_1Q_1+\Phi_2Q_2\right)\right)$$

But $C=\left(Q_1+Q_2\right)\left(\nabla_1\left(Q_1+Q_2\right)-\left(\Phi_1Q_1+\Phi_2Q_2\right)\right)$ can be more simplified as follows

$$\begin{aligned}
&\left(Q_1+Q_2\right)\left(\nabla_1\left(Q_1+Q_2\right)-\left(\Phi_1Q_1+\Phi_2Q_2\right)\right)\\
&=\nabla_1(Q_1+Q_2)^2\left(1-\frac{\left(\Phi_1Q_1+\Phi_2Q_2\right)}{\nabla_1\left(Q_1+Q_2\right)}\right)\\
&=\nabla_1(Q_1+Q_2)^2\left(1-\left[\frac{\Phi_1Q_1}{\nabla_1\left(Q_1+Q_2\right)}+\frac{\Phi_2Q_2}{\nabla_1\left(Q_1+Q_2\right)}\right]\right)\\
&=\nabla_1(Q_1+Q_2)^2\left(1-\left[\frac{\left(\beta_1+\beta_2\omega_1+\beta_3\omega_2\right)Q_1}{\nabla_1\left(Q_1+Q_2\right)}+\frac{\Phi_1Q_1+\Phi_2Q_2}{\nabla_1\left(Q_1+Q_2\right)}\right]\right)\\
&=\nabla_1(Q_1+Q_2)^2\left(1-\left[\frac{\left(\beta_1+\beta_2\omega_1+\beta_3\omega_2\right)Q_1}{\nabla_1\left(Q_1+Q_2\right)}+\frac{\Phi_2}{\nabla_1}\right]\right)\\
&=\nabla_1(Q_1+Q_2)^2\left(1-R_0\right)
\end{aligned}$$

Thus ${I_1}^*=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$

$$=\frac{-(Q_1+Q_2)(\Phi_1\nabla_1+\Phi_2\nabla_1-\Phi_1\Phi_2)\pm\sqrt{((Q_1+Q_2)(\Phi_1\nabla_1+\Phi_2\nabla_1-\Phi_1\Phi_2))^2-4\Phi_1\Phi_2\nabla_1\nabla_1(Q_1+Q_2)^2(1-R_0)}}{2\Phi_1\Phi_2\nabla_1}$$

Here we have the following cases.

Case i) for $R_0>1$ we have $C<0$ in this case the quadratic equation $\mathrm{A}{I_1}^{*2}+B{I_1}^*+C=0$ has two roots with opposite sign. The negative root is biologically meaningless, hence the uniqueness of the endemic equilibrium.

Case ii) for $R_0>1$ and $\Phi_1\nabla_1+\Phi_2\nabla_1-\Phi_1\Phi_2<0$ the quadratic equation $\mathrm{A}{I_1}^{*2}+B{I_1}^*+C=0$ has two endemic equilibriums.

Case iii) for $R_0<1$ and $\Phi_1\nabla_1+\Phi_2\nabla_1-\Phi_1\Phi_2<0$ the quadratic equation $\mathrm{A}{I_1}^{*2}+B{I_1}^*+C=0$ has two endemic equilibriums. Case iv) for $R_0=1$ the model has two equilibriums, namely,

the disease-free equilibrium point and the endemic equilibrium point, coexisting.

Thus we get the endemic equilibrium point

$E^* = (S_1^*, S_2^*, I_1^*, I_2^*, P^*, S_p^*, A^*)$ where

$$I_1^* = \frac{-(Q_1+Q_2)(\Phi_1\nabla_1 + \Phi_2\nabla_1 - \Phi_1\Phi_2)}{2\Phi_1\Phi_2\nabla_1}$$

$$\pm\frac{\sqrt{((Q_1+Q_2)(\Phi_1\nabla_1 + \Phi_2\nabla_1 - \Phi_1\Phi_2))^2 - 4\Phi_1\Phi_2\nabla_1^2(Q_1+Q_2)^2(1-R_0)}}{2\Phi_1\Phi_2\nabla_1}$$

$$S_1^* = \frac{Q_1}{(\frac{(\Phi_1 I_1^*)}{N^*} + \mu)}, S_2^* = \frac{Q_2}{(\frac{(\Phi_2 I_1^*)}{N^*} + \mu)}, I_2^* = \frac{\theta}{\nabla_2} I_1^*, P^* = \frac{(u_1\nabla_2 + u_2\theta)}{\nabla_2\nabla_3} I_1^*,$$

$$S_p^* = \frac{k_1 + k_2\frac{\theta}{\nabla_2} + k_3\frac{(u_1\nabla_2+u_2\theta)}{\nabla_2\nabla_3}}{\mu} I_1^* \quad and \quad A^* = \frac{\delta_1 + \delta_2\frac{\theta}{\nabla_2} + \delta_3\frac{(u_1\nabla_2+u_2\theta)}{\nabla_2\nabla_3}}{(\alpha+\mu)} I_1^*$$

$$\Phi_1 = (\beta_1+\sigma) + (\beta_2+\sigma)\frac{\theta}{\nabla_2} + (\beta_3+\sigma)\frac{(u_1\nabla_2+u_2\theta)}{\nabla_2\nabla_3}, \Phi_2 = \sigma(1 + \frac{\theta}{\nabla_2} + \frac{u_1\nabla_2+u_2\theta)}{\nabla_2\nabla_3})$$

**Local stability of endemic equilibrium point**

We now investigate the local stability of the endemic equilibrium point $E^*$.

**Theorem 5.5.** *The positive endemic equilibrium point $E^*$ of the system of equations (5.1) − (5.7) is locally asymptotically stable if $R_0 > 1$.*

*Proof.* the linearization of the Jacobian matrix of the system of equations (5.1) − (5.7) at any point is

$$J(E) = \begin{pmatrix} M_{11} & 0 & M_{13} & M_{14} & M_{15} & 0 & 0 \\ 0 & M_{22} & M_{23} & M_{24} & M_{25} & 0 & 0 \\ \Lambda_1 & \Lambda_2 & M_{33}-\nabla_1 & M_{34} & M_{35} & 0 & 0 \\ 0 & 0 & \theta & -\nabla_2 & 0 & 0 & 0 \\ 0 & 0 & u_1 & u_2 & -\nabla_3 & 0 & 0 \\ 0 & 0 & k_1 & k_2 & k_3 & -\mu & 0 \\ 0 & 0 & \delta_1 & \delta_2 & \delta_3 & 0 & -(\alpha+\mu) \end{pmatrix}$$

where $M_{11} = -(\Lambda_1+\mu)$, $M_{13} = -\left(\frac{\beta_1+\sigma}{N}\right)S_1$, $M_{14} = -\left(\frac{\beta_2+\sigma}{N}\right)S_1$, $M_{15} = -\left(\frac{\beta_3+\sigma}{N}\right)S_1$

$M_{22} = -(\Lambda_2+\mu)$, $M_{23} = -\sigma\frac{S_2}{N}$, $M_{24} = -\sigma\frac{S_2}{N}$, $M_{25} = -\sigma\frac{S_2}{N}$,

$M_{33} = \left(\frac{\beta_1+\sigma}{N}\right)S_1 + \sigma\frac{S_2}{N}$ , $M_{34} = \left(\frac{\beta_2+\sigma}{N}\right)S_1 + \sigma\frac{S_2}{N}$, $M_{35} = \left(\frac{\beta_3+\sigma}{N}\right)S_1 + \sigma\frac{S_2}{N}$

At the endemic equilibrium point the above Jacobian matrix becomes

$$J(E^*) = \begin{bmatrix} M_{11}^* & 0 & M_{13}^* & M_{14}^* & M_{15}^* & 0 & 0 \\ 0 & M_{22}^* & M_{23}^* & M_{24}^* & M_{25}^* & 0 & 0 \\ \Lambda_1^* & \Lambda_2^* & M_{33}^* - \nabla_1 & M_{34}^* & M_{35}^* & 0 & 0 \\ 0 & 0 & \theta & -\nabla_2 & 0 & 0 & 0 \\ 0 & 0 & u_1 & u_2 & -\nabla_3 & 0 & 0 \\ 0 & 0 & k_1 & k_2 & k_3 & -\mu & 0 \\ 0 & 0 & \delta_1 & \delta_2 & \delta_3 & 0 & -(\alpha+\mu)) \end{bmatrix}$$

where $M_{11}^* = -(\Lambda_1^* + \mu), M_{13}^* = -(\frac{\beta_1+\sigma}{N^*})S_1^*, M_{14}^* = -(\frac{\beta_2+\sigma}{N^*})S_1^*, M_{15}^* = -(\frac{\beta_3+\sigma}{N^*})S_1^*$

$M_{22}^* = -(\Lambda_2 + \mu), M_{23}^* = -\sigma\frac{S_2^*}{N^*}, M_{24}^* = -\sigma\frac{S_2^*}{N^*}, M_{25}^* = -\sigma\frac{S_2^*}{N^*},$

$M_{33}^* = (\frac{\beta_1+\sigma}{N^*})S_1^* + \sigma\frac{S_2^*}{N^*}, M_{34}^* = (\frac{\beta_2+\sigma)}{N^*})S_1^* + \sigma\frac{S_2^*}{N^*}, M_{35}^* = (\frac{\beta_3+\sigma}{N^*})S_1^* + \sigma\frac{S_2^*}{N^*}$

The corresponding characteristic equation is

$$\begin{vmatrix} M_{11}^* - \lambda & 0 & M_{13}^* & M_{14}^* & M_{15}^* & 0 & 0 \\ 0 & M_{22}^* - \lambda & M_{23}^* & M_{24}^* & M_{25}^* & 0 & 0 \\ \Lambda_1^* & \Lambda_2^* & M_{33}^* - \nabla_1 - \lambda & M_{34}^* & M_{35}^* & 0 & 0 \\ 0 & 0 & \theta & -\nabla_2 - \lambda & 0 & 0 & 0 \\ 0 & 0 & u_1 & u_2 & -\nabla_3 - \lambda & 0 & 0 \\ 0 & 0 & k_1 & k_2 & k_3 & -\mu - \lambda & 0 \\ 0 & 0 & \delta_1 & \delta_2 & \delta_3 & 0 & -(\alpha+\mu) - \lambda) \end{vmatrix} = 0$$

$$\Rightarrow ((\alpha+\mu)+\lambda)(\mu+\lambda) \begin{vmatrix} M_{11}^* - \lambda & 0 & M_{13}^* & M_{14}^* & M_{15}^* \\ 0 & M_{22}^* - \lambda & M_{23}^* & M_{24}^* & M_{25}^* \\ \Lambda_1^* & \Lambda_2^* & M_{33}^* - (\nabla_1 + \lambda) & M_{34}^* & M_{35}* \\ 0 & 0 & \theta & -\nabla_2 - \lambda & 0 \\ 0 & 0 & u_1 & u_2 & -\nabla_3 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = -(\alpha+\mu) \, or \lambda = -\mu \; or$$

$$\begin{vmatrix} M_{11}^* - \lambda & 0 & M_{13}^* & M_{14}^* & M_{15}^* \\ 0 & M_{22}^* - \lambda & M_{23}^* & M_{24}^* & M_{25}^* \\ \Lambda_1^* & \Lambda_2^* & M_{33}^* - (\nabla_1 + \lambda) & M_{34}^* & M_{35}^* \\ 0 & 0 & \theta & -\nabla_2 - \lambda & 0 \\ 0 & 0 & u_1 & u_2 & -\nabla_3 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (M_{11}^*-\lambda)\begin{vmatrix} M_{22}^*-\lambda & M_{23}^* & M_{24}^* & M_{25}^* \\ \Lambda_2^* & M_{33}^*-(\nabla_1+\lambda) & M_{34}^* & M_{35}^* \\ 0 & \theta & -\nabla_2-\lambda & 0 \\ 0 & u_1 & u_2 & -\nabla_3-\lambda \end{vmatrix}$$

$$+\Lambda_1^*\begin{vmatrix} 0 & M_{13}^* & M_{14}^* & M_{15}^* \\ M_{22}^*-\lambda & M_{23}^* & M_{24}^* & M_{25}^* \\ 0 & \theta & -\nabla_2-\lambda & 0 \\ 0 & u_1 & u_2 & -\nabla_3-\lambda \end{vmatrix}=0$$

$$\Rightarrow (M_{11}^*-\lambda)(M_{22}^*-\lambda)\begin{vmatrix} M_{33}^*-(\nabla_1+\lambda) & M_{34}^* & M_{35}^* \\ \theta & -\nabla_2-\lambda & 0 \\ u_1 & u_2 & -\nabla_3-\lambda \end{vmatrix}$$

$$-\Lambda_2^*(M_{11}^*-\lambda)\begin{vmatrix} M_{23}^* & M_{24}^* & M_{25}^* \\ \theta & -\nabla_2-\lambda & 0 \\ u_1 & u_2 & -\nabla_3-\lambda \end{vmatrix}-\Lambda_1^*(M_{22}^*-\lambda)\begin{vmatrix} M_{13}^* & M_{14}^* & M_{15}^* \\ \theta & -\nabla_2-\lambda & 0 \\ u_1 & u_2 & -\nabla_3-\lambda \end{vmatrix}=0$$

$$\Rightarrow (M_{11}^*-\lambda)(M_{22}^*-\lambda)\left[M_{35}^*\begin{vmatrix}\theta & -\nabla_2-\lambda \\ u_1 & u_2\end{vmatrix}-(\nabla_3+\lambda)\begin{vmatrix} M_{33}^*-(\nabla_1+\lambda) & M_{34}^* \\ \theta & -\nabla_2-\lambda\end{vmatrix}\right]$$

$$-\Lambda_2^*(M_{11}^*-\lambda)\left[M_{25}^*\begin{vmatrix}\theta & -\nabla_2-\lambda \\ u_1 & u_2\end{vmatrix}-(\nabla_3+\lambda)\begin{vmatrix} M_{23}^* & M_{24}^* \\ \theta & -\nabla_2-\lambda\end{vmatrix}\right]$$

$$-\Lambda_1^*(M_{22}^*-\lambda)\left[M_{15^*}\begin{vmatrix}\theta & -\nabla_2-\lambda \\ u_1 & u_2\end{vmatrix}-(\nabla_3+\lambda)\begin{vmatrix} M_{13}^* & M_{14}^* \\ \theta & -\nabla_2-\lambda\end{vmatrix}\right]=0$$

$$\Rightarrow (M_{11}^*-\lambda)(M_{22}^*-\lambda)M_{35}^*\begin{vmatrix}\theta & -\nabla_2-\lambda \\ u_1 & u_2\end{vmatrix}$$

$$-(M_{11}^*-\lambda)(M_{22}^*-\lambda)(\nabla_3+\lambda)\begin{vmatrix} M_{33}^*-(\nabla_1+\lambda) & M_{34}^* \\ \theta & -\nabla_2-\lambda\end{vmatrix}$$

$$-\Lambda_2^*(M_{11}^*-\lambda)M_{25}^*\begin{vmatrix}\theta & -\nabla_2-\lambda \\ u_1 & u_2\end{vmatrix}+\Lambda_2^*(M_{11}^*-\lambda)(\nabla_3+\lambda)\begin{vmatrix} M_{23}^* & M_{24}^* \\ \theta & -\nabla_2-\lambda\end{vmatrix}$$

$$-\Lambda_1^*(M_{22}^*-\lambda)M_{15}^*\begin{vmatrix}\theta & -\nabla_2-\lambda \\ u_1 & u_2\end{vmatrix}+\Lambda_1^*(M_{22}^*-\lambda)(\nabla_3+\lambda)\begin{vmatrix} M_{13}^* & M_{14}^* \\ \theta & -\nabla_2-\lambda\end{vmatrix}=0$$

$$\Rightarrow [(M_{11}^*-\lambda)(M_{22}^*-\lambda)[M_{35}^*(\theta u_2+u_1(\nabla_2+\lambda))+(M_{33}^*-(\nabla_1+\lambda)(\nabla_2+\lambda)+\theta M_{34}^*)(\nabla_3+\lambda)]$$

$$-\Lambda_2^*(M_{11}-\lambda)[M_{25}(\theta u_2+u_1(\nabla_2+\lambda))+(M_{23}(\nabla_2+\lambda)+\theta M_{24})(\nabla_3+\lambda)]$$

$$-\Lambda_1^*(M_{22}^*-\lambda)[M_{15}^*(\theta u_2+u_1(\nabla_2+\lambda))+(M_{13}^*(\nabla_2+\lambda)+\theta M_{14}^*)(\nabla_3+\lambda)]=0$$

$$\Rightarrow (M_{11}^*-\lambda)(M_{22}^*-\lambda)[M_{35}^*\theta u_2+M_{35}^*u_1\nabla_2+M_{35}^*u_1\lambda]$$

$$+(M_{11}^*-\lambda)(M_{22}^*-\lambda)(M_{33}^*\nabla_2+M_{33}^*\lambda-\nabla_1\nabla_2-(\nabla_1+\nabla_2)\lambda-\lambda^2+\theta M_{34}^*)(\nabla_3+\lambda)$$

$$
\begin{aligned}
&-\Lambda_2^*\left(M_{11}^*-\lambda\right)\left[M_{25}^*\theta u_2+M_{25}^*u_1\nabla_2+M_{25}^*u_1\lambda+\left(M_{23}^*\nabla_2+M_{23}^*\lambda+\theta M_{24}^*\right)\left(\nabla_3+\lambda\right)\right]\\
&-\Lambda_1^*\left(M_{22}^*-\lambda\right)\left[M_{15}^*\theta u_2+M_{15}^*u_1\nabla_2+M_{15}^*u_1\lambda+\left(M_{13}^*\nabla_2+M_{13}^*\lambda+\theta M_{14}^*\right)\left(\nabla_3+\lambda\right)\right]=0\\
\Rightarrow&\left(M_{11}^*M_{22}^*-\left(M_{11}^*+M_{22}^*\right)\lambda+\lambda^2\right)\left[M_{35}^*\left(\theta u_2+u_1\nabla_2\right)+M_{35}^*u_1\lambda+M_{33}^*\nabla_2\nabla_3+M_{33}^*\nabla_3\lambda\right]\\
&+\left(M_{11}^*M_{22}^*-\left(M_{11}^*+M_{22}^*\right)\lambda+\lambda^2\right)\left[-\nabla_1\nabla_2\nabla_3-\nabla_2\nabla_3\lambda-\nabla_1\nabla_3\lambda-\nabla_3\lambda^2+\theta M_{34}^*\nabla_3\right]\\
&+\left(M_{11}^*M_{22}^*-\left(M_{11}^*+M_{22}^*\right)\lambda+\lambda^2\right)\left[M_{33}^*\nabla_2\lambda+M_{33}^*\lambda^2-\nabla_1\nabla_2\lambda-\nabla_2\lambda^2-\nabla_1\lambda^2\right]\\
&+\left(M_{11}^*M_{22}^*-\left(M_{11}^*+M_{22}^*\right)\lambda+\lambda^2\right)\left[-\lambda^3+\theta M_{34}^*\lambda\right]\\
&-\Lambda_2^*\left(M_{11}^*-\lambda\right)\left[M_{25}^*\left(\theta u_2+u_1\nabla_2\right)+M_{23}^*\nabla_2\nabla_3+M_{25}^*u_1\lambda+M_{23}^*\nabla_3\lambda+M_{23}^*\nabla_2\lambda+\theta M_{24}^*\nabla_3\right]\\
&-\Lambda_2^*\left(M_{11}^*-\lambda\right)\left[M_{23}^*\lambda^2+\theta M_{24}^*\lambda\right]-\Lambda_1^*\left(M_{22}^*-\lambda\right)\left[M_{15}^*\left(\theta u_2+u_1\nabla_2\right)+M_{13}^*\nabla_2\nabla_3\right]\\
&-\Lambda_1^*\left(M_{22}^*-\lambda\right)\left[M_{15}^*u_1\lambda+M_{13}^*\nabla_3\lambda+M_{13}^*\nabla_2\lambda+\theta M_{14}^*\nabla_3+M_{13}^*\lambda^2+\theta M_{14}^*\lambda\right]=0\\
\Rightarrow&\left(M_{11}^*M_{22}^*-\left(M_{11}^*+M_{22}^*\right)\lambda+\lambda^2\right)\left[-\lambda^3+\left(M_{33}^*-\nabla_1-\nabla_2-\nabla_3\right)\lambda^2+\left(M_{35}^*u_1+M_{33}^*\nabla_3\right.\right.\\
&\left.+M_{33}^*\nabla_2+\theta M_{34}^*-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2\right)\lambda+M_{35}^*\left(\theta u_2+u_1\nabla_2\right)+M_{33}^*\nabla_2\nabla_3\\
&\left.-\nabla_1\nabla_2\nabla_3+\theta M_{34}^*\nabla_3\right]-\Lambda_2^*\left(M_{11}^*-\lambda\right)\left[M_{23}\lambda^2+\left(M_{25}u_1+M_{23}\nabla_3+M_{23}^*\nabla_2+\theta M_{24}^*\right)\lambda\right]\\
&-\Lambda_2^*\left(M_{11}^*-\lambda\right)\left[M_{25}^*\left(\theta u_2+u_1\nabla_2\right)+M_{23}^*\nabla_2\nabla_3+\theta M_{24}^*\nabla_3\right]-\Lambda_1^*\left(M_{22}^*-\lambda\right)\left[M_{13}^*\lambda^2+\left(M_{15}^*u_1\right.\right.\\
&\left.-\Lambda_1^*\left(M_{22}^*-\lambda\right)\left[M_{13}^*\nabla_3+M_{13}^*\nabla_2+\theta M_{14}^*\right)\lambda+M_{15}^*\left(\theta u_2+u_1\nabla_2\right)+M_{13}^*\nabla_2\nabla_3+\theta M_{14}^*\nabla_3\right]=0\\
\Rightarrow&-\lambda^3\left(M_{11}^*M_{22}^*-\left(M_{11}^*+M_{22}^*\right)\lambda+\lambda^2\right)+\left(M_{11}^*M_{22}^*-\left(M_{11}^*+M_{22}^*\right)\lambda+\lambda^2\right)\left(M_{33}^*-\nabla_1-\nabla_2\right.\\
&\left.-\nabla_3\right)\lambda^2+\left(M_{11}^*M_{22}^*-\left(M_{11}^*+M_{22}^*\right)\lambda+\lambda^2\right)\left(M_{35}^*u_1+M_{33}^*\nabla_3+M_{33}^*\nabla_2+\theta M_{34}^*\right.\\
&\left.-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2\right)\lambda+\left(M_{11}^*M_{22}^*-\left(M_{11}^*+M_{22}^*\right)\lambda+\lambda^2\right)\left(M_{35}^*\left(\theta u_2+u_1\nabla_2\right)\right.\\
&\left.+M_{33}^*\nabla_2\nabla_3-\nabla_1\nabla_2\nabla_3+\theta M_{34}^*\nabla_3\right)-\Lambda_2^*\left(M_{11}^*-\lambda\right)M_{23}^*\lambda^2-\Lambda_2^*\left(M_{11}^*-\lambda\right)\left(M_{25}^*u_1\right.\\
&\left.+M_{23}^*\nabla_3+M_{23}^*\nabla_2+\theta M_{24}^*\right)\lambda-\Lambda_2^*\left(M_{11}^*-\lambda\right)M_{25}^*\left(\theta u_2+u_1\nabla_2\right)-\Lambda_2^*\left(M_{11}^*-\lambda\right)\left(M_{23}^*\nabla_2\nabla_3\right.\\
&\left.-\Lambda_2^*\left(M_{11}^*-\lambda\right)\theta M_{24}^*\nabla_3-\Lambda_1^*\left(M_{22}^*-\lambda\right)M_{13}^*\lambda^2-\Lambda_1^*\left(M_{22}^*-\lambda\right)\left(M_{15}^*u_1+M_{13}^*\nabla_3\right.\right.\\
&\left.-\Lambda_1^*\left(M_{22}^*-\lambda\right)\left(M_{13}^*\nabla_2+\theta M_{14}^*\right)\lambda-\Lambda_1^*\left(M_{22}^*-\lambda\right)M_{15}^*\left(\theta u_2+u_1\nabla_2\right)\right.\\
&\left.-\Lambda_1^*\left(M_{22}^*-\lambda\right)\left(M_{13}^*\nabla_2\nabla_3+\theta M_{14}^*\nabla_3\right)=0\right.\\
\Rightarrow&-\lambda^3M_{11}^*M_{22}^*+\left(M_{11}^*+M_{22}^*\right)\lambda^4-\lambda^5+M_{11}^*M_{22}^*\left(M_{33}^*-\nabla_1-\nabla_2-\nabla_3\right)\lambda^2\\
&-\left(M_{11}^*+M_{22}^*\right)\left(M_{33}^*-\nabla_1-\nabla_2-\nabla_3\right)\lambda^3+\left(M_{33}^*-\nabla_1-\nabla_2-\nabla_3\right)\lambda^4\\
&+M_{11}^*M_{22}^*(M_{35}^*u_1+M_{33}^*\nabla_3+M_{33}^*\nabla_2+\theta M_{34}^*-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2)\lambda\\
&-\left(M_{11}^*+M_{22}^*\right)\left(M_{35}^*u_1+M_{33}^*\nabla_3+M_{33}^*\nabla_2+\theta M_{34}^*-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2\right)\lambda^2\\
&+\left(M_{35}^*u_1+M_{33}^*\nabla_3+M_{33}^*\nabla_2+\theta M_{34}^*-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2\right)\lambda^3\\
&+M_{11}^*M_{22}^*\left(M_{35}^*\left(\theta u_2+u_1\nabla_2\right)+M_{33}^*\nabla_2\nabla_3-\nabla_1\nabla_2\nabla_3+\theta M_{34}^*\nabla_3\right)\\
&-\left(M_{11}^*+M_{22}^*\right)\left(M_{35}^*\left(\theta u_2+u_1\nabla_2\right)+M_{33}^*\nabla_2\nabla_3-\nabla_1\nabla_2\nabla_3+\theta M_{34}^*\nabla_3\right)\lambda\\
&+\left(M_{35}^*\left(\theta u_2+u_1\nabla_2\right)+M_{33}^*\nabla_2\nabla_3-\nabla_1\nabla_2\nabla_3+\theta M_{34}^*\nabla_3\right)\lambda^2-\Lambda_2^*M_{11}^*M_{23}^*\lambda^2\\
&+\Lambda_2^*M_{23}^*\lambda^3-\Lambda_2^*M_{11}^*\left(M_{25}^*u_1+M_{23}^*\nabla_3+M_{23}^*\nabla_2+\theta M_{24}^*\right)\lambda+\Lambda_2^*(M_{25}^*u_1+M_{23}^*\nabla_3\\
&+M_{23}^*\nabla_2+\theta M_{24}^*)\lambda^2-\Lambda_2^*M_{11}^*M_{25}^*\left(\theta u_2+u_1\nabla_2\right)+\Lambda_2^*M_{25}^*\left(\theta u_2+u_1\nabla_2\right)\lambda\\
&-\Lambda_2^*M_{11}^*\left(M_{23}^*\nabla_2\nabla_3+\theta M_{24}^*\nabla_3\right)+\Lambda_2^*\left(M_{23}^*\nabla_2\nabla_3+\theta M_{24}^*\nabla_3\right)\lambda-\Lambda_1^*M_{22}^*M_{13}^*\lambda^2\\
&+\Lambda_1^*M_{13}^*\lambda^3-\Lambda_1^*M_{22}^*\left(M_{15}^*u_1+M_{13}^*\nabla_3+M_{13}^*\nabla_2+\theta M_{14}^*\right)\lambda+\Lambda_1^*(M_{15}^*u_1+M_{13}^*\nabla_3
\end{aligned}
$$

$$
\begin{aligned}
&+M_{13}^{*}\nabla_2+\theta M_{14}^{*})\lambda^2-\Lambda_1^{*}M_{22}^{*}M_{15}^{*}\left(\theta u_2+u_1\nabla_2\right)+\Lambda_1^{*}M_{15}^{*}\left(\theta u_2+u_1\nabla_2\right)\lambda\\
&-\Lambda_1^{*}M_{22}^{*}\left(M_{13}^{*}\nabla_2\nabla_3+\theta M_{14}^{*}\nabla_3\right)+\Lambda_1^{*}\left(M_{13}^{*}\nabla_2\nabla_3+\theta M_{14}^{*}\nabla_3\right)\lambda=0\\
\Rightarrow\ &-\lambda^5+\left(M_{11}^{*}+M_{22}^{*}+M_{33}^{*}-\nabla_1-\nabla_2-\nabla_3\right)\lambda^4+[-M_{11}^{*}M_{22}^{*}\\
&-\left(M_{11}^{*}+M_{22}^{*}\right)\left(M_{33}^{*}-\nabla_1-\nabla_2-\nabla_3\right)+\Lambda_2^{*}M_{23}^{*}+\Lambda_1^{*}M_{13}^{*}+\ \left(M_{35}^{*}u_1+M_{33}^{*}\nabla_3\right.\\
&\left.+M_{33}^{*}\nabla_2+\theta M_{34}^{*}-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2\right)]\lambda^3+[\,M_{11}^{*}M_{22}^{*}\left(M_{33}^{*}-\nabla_1-\nabla_2-\nabla_3\right)\\
&-\left(M_{11}^{*}+M_{22}^{*}\right)\left(M_{35}^{*}u_1+M_{33}^{*}\nabla_3+M_{33}^{*}\nabla_2+\theta M_{34}^{*}-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2\right)\\
&+M_{35}^{*}\left(\theta u_2+u_1\nabla_2\right)+M_{33}^{*}\nabla_2\nabla_3-\nabla_1\nabla_2\nabla_3+\theta M_{34}^{*}\nabla_3-\Lambda_2^{*}M_{11}^{*}M_{23}^{*}+\Lambda_2^{*}(M_{25}^{*}u_1\\
&+M_{23}^{*}\nabla_3+M_{23}^{*}\nabla_2+\theta M_{24}^{*})-\Lambda_1^{*}M_{22}^{*}M_{13}^{*}+\Lambda_1^{*}(M_{15}^{*}u_1+M_{13}^{*}\nabla_3+M_{13}^{*}\nabla_2\\
&+\theta M_{14}^{*})\,]\lambda^2+[\,M_{11}^{*}M_{22}^{*}(M_{35}^{*}u_1+M_{33}^{*}\nabla_3+M_{33}^{*}\nabla_2+\theta M_{34}^{*}-\nabla_2\nabla_3-\nabla_1\nabla_3\\
&-\nabla_1\nabla_2)-\left(M_{11}^{*}+M_{22}^{*}\right)\left(M_{35}^{*}\left(\theta u_2+u_1\nabla_2\right)+M_{33}^{*}\nabla_2\nabla_3-\nabla_1\nabla_2\nabla_3+\theta M_{34}^{*}\nabla_3\right)\\
&-\Lambda_2^{*}M_{11}^{*}\left(M_{25}^{*}u_1+M_{23}^{*}\nabla_3+M_{23}^{*}\nabla_2+\theta M_{24}^{*}\right)+\Lambda_2^{*}M_{25}^{*}\left(\theta u_2+u_1\nabla_2\right)\\
&+\Lambda_2^{*}\left(M_{23}^{*}\nabla_2\nabla_3+\theta M_{24}^{*}\nabla_3\right)-\Lambda_1^{*}M_{22}^{*}\left(M_{15}^{*}u_1+M_{13}^{*}\nabla_3+M_{13}^{*}\nabla_2+\theta M_{14}^{*}\right)\\
&+\Lambda_1^{*}M_{15}^{*}\left(\theta u_2+u_1\nabla_2\right)+\Lambda_1^{*}\left(M_{13}^{*}\nabla_2\nabla_3+\theta M_{14}^{*}\nabla_3\right)]\lambda\\
&+[-M_{11}^{*}M_{22}^{*}\left(M_{35}^{*}\left(\theta u_2+u_1\nabla_2\right)+M_{33}^{*}\nabla_2\nabla_3-\nabla_1\nabla_2\nabla_3+\theta M_{34}^{*}\nabla_3\right)\\
&+\Lambda_2^{*}M_{11}^{*}M_{25}^{*}\left(\theta u_2+u_1\nabla_2\right)+\Lambda_2^{*}M_{11}^{*}\left(M_{23}^{*}\nabla_2\nabla_3+\theta M_{24}^{*}\nabla_3\right)\\
&+\Lambda_1^{*}M_{22}^{*}M_{15}^{*}\left(\theta u_2+u_1\nabla_2\right)+\Lambda_1^{*}M_{22}^{*}\left(M_{13}^{*}\nabla_2\nabla_3+\theta M_{14}^{*}\nabla_3\right)]=0\\
\Rightarrow\ &\lambda^5+\left(\nabla_1+\nabla_2+\nabla_3-M_{11}^{*}-M_{22}^{*}-M_{33}^{*}\right)\lambda^4+[M_{11}^{*}M_{22}^{*}\\
&+\left(M_{11}^{*}+M_{22}^{*}\right)\left(M_{33}^{*}-\nabla_1-\nabla_2-\nabla_3\right)-\Lambda_2^{*}M_{23}^{*}-\Lambda_1^{*}M_{13}^{*}-\ \left(M_{35}^{*}u_1+M_{33}^{*}\nabla_3\right.\\
&\left.+M_{33}^{*}\nabla_2+\theta M_{34}^{*}-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2\right)]\lambda^3+[\,-M_{11}^{*}M_{22}^{*}\left(M_{33}^{*}-\nabla_1-\nabla_2-\nabla_3\right)\\
&+\left(M_{11}^{*}+M_{22}^{*}\right)\left(M_{35}^{*}u_1+M_{33}^{*}\nabla_3+M_{33}^{*}\nabla_2+\theta M_{34}^{*}-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2\right)\\
&-M_{35}^{*}\left(\theta u_2+u_1\nabla_2\right)-M_{33}^{*}\nabla_2\nabla_3+\nabla_1\nabla_2\nabla_3-\theta M_{34}^{*}\nabla_3+\Lambda_2^{*}M_{11}^{*}M_{23}^{*}-\Lambda_2^{*}(M_{25}^{*}u_1\\
&+M_{23}^{*}\nabla_3+M_{23}^{*}\nabla_2+\theta M_{24}^{*})+\Lambda_1^{*}M_{22}^{*}M_{13}^{*}-\Lambda_1^{*}\left(M_{15}^{*}u_1+M_{13}^{*}\nabla_3+M_{13}^{*}\nabla_2+\theta M_{14}^{*}\right)]\lambda^2\\
&+[-M_{11}^{*}M_{22}^{*}(M_{35}^{*}u_1+M_{33}^{*}\nabla_3+M_{33}^{*}\nabla_2+\theta M_{34}^{*}-\nabla_2\nabla_3-\nabla_1\nabla_3-\nabla_1\nabla_2)\\
&+\left(M_{11}^{*}+M_{22}^{*}\right)\left(M_{35}^{*}\left(\theta u_2+u_1\nabla_2\right)+M_{33}^{*}\nabla_2\nabla_3-\nabla_1\nabla_2\nabla_3+\theta M_{34}^{*}\nabla_3\right)+\Lambda_2^{*}M_{11}^{*}(M_{25}^{*}u_1\\
&+M_{23}^{*}\nabla_3+M_{23}^{*}\nabla_2+\theta M_{24}^{*})-\Lambda_2^{*}M_{25}^{*}\left(\theta u_2+u_1\nabla_2\right)-\Lambda_2^{*}\left(M_{23}^{*}\nabla_2\nabla_3+\theta M_{24}^{*}\nabla_3\right)\\
&+\Lambda_1^{*}M_{22}^{*}\left(M_{15}^{*}u_1+M_{13}^{*}\nabla_3+M_{13}^{*}\nabla_2+\theta M_{14}^{*}\right)-\Lambda_1^{*}M_{15}^{*}\left(\theta u_2+u_1\nabla_2\right)-\Lambda_1^{*}(M_{13}^{*}\nabla_2\nabla_3\\
&+\theta M_{14}^{*}\nabla_3)\,]\lambda+[\,-M_{11}^{*}M_{22}^{*}\left(M_{35}^{*}\left(\theta u_2+u_1\nabla_2\right)+M_{33}^{*}\nabla_2\nabla_3-\nabla_1\nabla_2\nabla_3+\theta M_{34}^{*}\nabla_3\right)\\
&+\Lambda_2^{*}M_{11}^{*}M_{25}^{*}\left(\theta u_2+u_1\nabla_2\right)+\Lambda_2^{*}M_{11}^{*}\left(M_{23}^{*}\nabla_2\nabla_3+\theta M_{24}^{*}\nabla_3\right)\\
&+\Lambda_1^{*}M_{22}^{*}M_{15}^{*}\left(\theta u_2+u_1\nabla_2\right)+\Lambda_1^{*}M_{22}^{*}\left(M_{13}^{*}\nabla_2\nabla_3+\theta M_{14}^{*}\nabla_3\right)]=0
\end{aligned}
$$

Then the above characteristic equation is given by

$$P\left(\lambda\right)=\lambda^5+a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5=0$$

Where

$a_1=\left(\nabla_1+\nabla_2+\nabla_3-M_{11}^{*}-M_{22}^{*}-M_{33}^{*}\right)$

$a_2 = [M_{11}^* M_{22}^* + (M_{11}^* + M_{22}^*)(M_{33}^* - \nabla_1 - \nabla_2 - \nabla_3) - \Lambda_2^* M_{23}^* - \Lambda_1^* M_{13}^* - (M_{35}^* u_1 + M_{33}^* \nabla_3 + M_{33}^* \nabla_2 + \theta M_{34}^* - \nabla_2\nabla_3 - \nabla_1\nabla_3 - \nabla_1\nabla_2)]$

$a_3 = [-M_{11}^* M_{22}^* (M_{33}^* - \nabla_1 - \nabla_2 - \nabla_3) + (M_{11}^* + M_{22}^*)(M_{35}^* u_1 + M_{33}^* \nabla_3 + M_{33}^* \nabla_2 + \theta M_{34}^* - \nabla_2\nabla_3 - \nabla_1\nabla_3 - \nabla_1\nabla_2) - M_{35}^*(\theta u_2 + u_1\nabla_2) - M_{33}^* \nabla_2\nabla_3 + \nabla_1\nabla_2\nabla_3 - \theta M_{34}^* \nabla_3 + \Lambda_2^* M_{11}^* M_{23}^* - \Lambda_2^*(M_{25}^* u_1 + M_{23}^* \nabla_3 + M_{23}^* \nabla_2 + \theta M_{24}^*) + \Lambda_1^* M_{22}^* M_{13}^* - \Lambda_1^*(M_{15}^* u_1 + M_{13}^* \nabla_3 + M_{13}^* \nabla_2 + \theta M_{14}^*)]$

$a_4 = [-M_{11}^* M_{22}^* (M_{35}^* u_1 + M_{33}^* \nabla_3 + M_{33}^* \nabla_2 + \theta M_{34}^* - nabla_2\nabla_3 - \nabla_1\nabla_3 - \nabla_1\nabla_2) + (M_{11}^* + M_{22}^*)(M_{35}^*(\theta u_2 + u_1\nabla_2) + M_{33}^* \nabla_2\nabla_3 - \nabla_1\nabla_2\nabla_3 + \theta M_{34}^* \nabla_3) + \Lambda_2^* M_{11}^*(M_{25}^* u_1 + M_{23}^* \nabla_3 + M_{23}^* \nabla_2 + \theta M_{24}^*) - \Lambda_2^* M_{25}^*(\theta u_2 + u_1\nabla_2) - \Lambda_2^*(M_{23}^* \nabla_2\nabla_3 + \theta M_{24}^* \nabla_3) + \Lambda_1^* M_{22}^*(M_{15}^* u_1 + M_{13}^* \nabla_3 + M_{13}^* \nabla_2 + \theta M_{14}^*) - \Lambda_1^* M_{15}^*(\theta u_2 + u_1\nabla_2) - \Lambda_1^*(M_{13}^* \nabla_2\nabla_3 + \theta M_{14}^* \nabla_3)]$

$a_5 = [-M_{11}^* M_{22}^*(M_{35}^*(\theta u_2 + u_1\nabla_2) + M_{33}^* \nabla_2\nabla_3 - \nabla_1\nabla_2\nabla_3 + \theta M_{34}^* \nabla_3) + \Lambda_2^* M_{11}^* M_{25}(\theta u_2 + u_1\nabla_2) + \Lambda_2^* M_{11}^*(M_{23}^* \nabla_2\nabla_3 + \theta M_{24}^* \nabla_3) + \Lambda_1^* M_{22}^* M_{15}^*(\theta u_2 + u_1\nabla_2) + \Lambda_1^* M_{22}^*(M_{13}^* \nabla_2\nabla_3 + \theta M_{14}^* \nabla_3)]$

Thus we can observe that

$a_1 > 0$ for $\nabla_1 + \nabla_2 + \nabla_3 - M_{11}^* - M_{22}^* > M_{33}^*$

$a_2 > 0$ for

$M_{11}^* M_{22}^* - (M_{11}^* + M_{22}^*)(\nabla_1 + \nabla_2 + \nabla_3) + \nabla_2\nabla_3 + \nabla_1\nabla_3 + \nabla_1\nabla_2 - \Lambda_2^* M_{23}^* - \Lambda_1^* M_{13}^* > (M_{35}^* u_1 + M_{33}^* \nabla_3 + M_{33}^* \nabla_2 + \theta M_{34}^*) - (M_{11}^* + M_{22}^*) M_{33}^*$

$a_3 > 0$ for

$M_{11}^* M_{22}^*(\nabla_1 + \nabla_2 + \nabla_3) - (M_{11}^* + M_{22}^*)(\nabla_2\nabla_3 + \nabla_1\nabla_3 + \nabla_1\nabla_2) + \nabla_1\nabla_2\nabla_3 - \Lambda_1^*(M_{15}^* u_1 + M_{13}^* \nabla_3 + M_{13}^* \nabla_2 + \theta M_{14}^*) - \Lambda_2^*(M_{25}^* u_1 + M_{23}^* \nabla_3 + M_{23}^* \nabla_2 + \theta M_{24}^*) + \Lambda_2^* M_{11}^* M_{23}^* + \Lambda_1^* M_{22}^* M_{13}^* > M_{11}^* M_{22}^* M_{33}^* - (M_{11}^* + M_{22}^*)(M_{35}^* u_1 + M_{33}^* \nabla_3 + M_{33}^* \nabla_2 + \theta M_{34}^*) + M_{35}^*(\theta u_2 + u_1\nabla_2) + M_{33}^* \nabla_2\nabla_3 + \theta M_{34}^* \nabla_3$

$a_4 > 0$ for

$M_{11}^* M_{22}^*(\nabla_2\nabla_3 + \nabla_1\nabla_3 + \nabla_1\nabla_2) - (M_{11}^* + M_{22}^*)\nabla_1\nabla_2\nabla_3 + \Lambda_2^* M_{11}^*(M_{25}^* u_1 + M_{23}^* \nabla_3 + M_{23}^* \nabla_2 + \theta M_{24}^*) - \Lambda_2^* M_{25}^*(\theta u_2 + u_1\nabla_2) - \Lambda_2^*(M_{23}^* \nabla_2\nabla_3 + \theta M_{24}^* \nabla_3) + \Lambda_1^* M_{22}^*(M_{15}^* u_1 + M_{13}^* \nabla_3 + M_{13}^* \nabla_2 + \theta M_{14}^*) - \Lambda_1^* M_{15}^*(\theta u_2 + u_1\nabla_2) - \Lambda_1^*(M_{13}^* \nabla_2\nabla_3 + \theta M_{14}^* \nabla_3) > M_{11}^* M_{22}^*(M_{35}^* u_1 + M_{33}^* \nabla_3 + M_{33}^* \nabla_2 + \theta M_{34}^*) - (M_{11}^* + M_{22}^*)(M_{35}^*(\theta u_2 + u_1\nabla_2) + M_{33}^* \nabla_2\nabla_3 + \theta M_{34}^* \nabla_3)$

$a_5 > 0$ for

$M_{11}^* M_{22}^* \nabla_1\nabla_2\nabla_3 + \Lambda_2^* M_{11}^* M_{25}^*(\theta u_2 + u_1\nabla_2) + \Lambda_2^* M_{11}^*(M_{23}^* \nabla_2\nabla_3 + \theta M_{24}^* \nabla_3) + \Lambda_1^* M_{22}^* M_{15}^*(\theta u_2 + u_1\nabla_2) + \Lambda_1^* M_{22}^*(M_{13}^* \nabla_2\nabla_3 + \theta M_{14}^* \nabla_3) > M_{11}^* M_{22}^*(M_{35}^*(\theta u_2 + u_1\nabla_2) + M_{33}^* \nabla_2\nabla_3 + \theta M_{34}^* \nabla_3)$

according to the above cases we can observe that all coefficients of the characteristic polynomial are positive.

To see the sign of eigenvalues we use Routh – Hurwitz criteria.

Consider the following Routh – Hurwitz array

| | | | |
|---|---|---|---|
| $\lambda^5$ | 1 | $a_2$ | $a_4$ |
| $\lambda^4$ | $a_1$ | $a_3$ | $a_5$ |
| $\lambda^3$ | $b_1$ | $b_2$ | |
| $\lambda^2$ | $c_1$ | $c_2$ | |
| $\lambda^1$ | $d_1$ | | |
| $\lambda^0$ | $e_1$ | | |

where $b_1 = \frac{a_1a_2-a_3}{a_1} > 0$, $b_2 = \frac{a_1a_4-a_5}{a_1} > 0$

$c_1 = \frac{b_1a_3-a_1b_2}{b_1} > 0$, $c_2 = a_5 > 0$

$d_1 = \frac{b_2c_1-b_1c_2}{c_1} > 0$, $e_1 = c_2 > 0$

Since all elements of the first column of the array have the same sign then by Routh – Hurwitz criteria all roots of the characteristic equation have negative real part, thus the endemic equilibrium point is locally asymptotically stable. □

**Global stability of endemic equilibrium point**

**Theorem 5.6.** *The endemic equilibrium point $E^*$ is globally asymptotically stable if $Z < Y$, where*

$$Z = \gamma_1(\theta I_1 + \nabla_2 I_2^*) + \gamma_2(u_1 I_1 + u_2 I_2 + \nabla_3 P^*) + \gamma_3(k_1 I_1 + k_2 I_2 + k_3 P + \mu S_p^*) + (\frac{\beta_1+\sigma}{N^*})I_1^* S_1$$
$$+(\frac{\beta_2+\sigma}{N^*})I_2^* S_1 + (\frac{\beta_3+\sigma}{N^*})P^* S_1 + (\frac{\beta_1+\sigma}{N})I_1 S_1^* + (\frac{\beta_2+\sigma}{N})I_2 S_1^* + (\frac{\beta_3+\sigma}{N})P S_1^* + \frac{\sigma}{N^*}I_1^* S_1 +$$
$$\frac{\sigma}{N^*}I_2^* S_1 + \frac{\sigma}{N^*}P^* S_2 + \frac{\sigma}{N}I_1 S_2^* + \frac{\sigma}{N}I_2 S_2^* + \frac{\sigma}{N}P S_2^*$$

*and*

$$Y = -[\mu S_1^*(2 - \frac{S_1}{S_1^*} - \frac{S_1^*}{S_1}) + \mu S_2^*(2 - \frac{S_2}{S_2^*} - \frac{S_2^*}{S_2}) + (2 - \frac{S_1}{S_1^*} - \frac{S_1^*}{S_1})(\frac{\beta_1+\sigma}{N^*})I_1^* S_1^* +$$
$$(2 - \frac{S_1}{S_1^*} - \frac{S_1^*}{S_1})(\frac{\beta_2+\sigma}{N^*})I_2^* S_1^* + (2 - \frac{S_1}{S_1^*} - \frac{S_1^*}{S_1})(\frac{\beta_3+\sigma}{N^*})P^* S_1^* +$$
$$(2 - \frac{S_2}{S_2^*} - \frac{S_2^*}{S_2})\frac{\sigma}{N^*}I_1^* S_2^* + (2 - \frac{S_2}{S_2^*} - \frac{S_2^*}{S_2})\frac{\sigma}{N^*}I_2^* S_2^* + (2 - \frac{S_2}{S_2^*} - \frac{S_2^*}{S_2})\frac{\sigma}{N^*}P^* S_2^*]$$
$$+[(\frac{\beta_1+\sigma}{N})I_1^* S_1 + (\frac{\beta_2+\sigma}{N})\frac{I_1^*}{I_1}I_2 S_1 + (\frac{\beta_3+\sigma}{N})\frac{I_1^*}{I_1}P S_1 + \frac{\sigma}{N}I_1^* S_2 + \frac{\sigma}{N}\frac{I_1^*}{I_1}I_2 S_2 + \frac{\sigma}{N}\frac{I_1^*}{I_1}P S_2 + \nabla_1 I_1 +$$
$$\gamma_1(\theta\frac{I_2^*}{I_2}I_1 + \nabla_2 I_2) + \gamma_2(u_1\frac{P^*}{P}I_1 + u_2\frac{P^*}{P}I_2 + \nabla_3 P) + \gamma_3(k_1\frac{S_p^*}{S_p}I_1 + k_2\frac{S_p^*}{S_p}I_2 + k_3\frac{S_p^*}{S_p}P + \mu S_p)]$$

*Proof.* Consider the following Lyapunov function

$V = (S_1 - S_1^* ln S_1) + (S_2 - S_2^* ln S_2) + (I_1 - I_1^* ln I_1) + \gamma_1(I_2 - I_2^* ln I_2) + \gamma_2(P - S_p^* ln P) + \gamma_3(S_p - S_p^* ln S_p)$

$+\gamma_4(A - A^* lnA)$

where $\gamma_i s$ for $i = 1, 2, 3, 4$ are non-negative quantities.

And thus we get $V$ is continuous function and has first order partial derivatives and $V$ has minimum at $E^*$.

$\frac{dV}{dt} = (1 - \frac{S_1^*}{S_1})\frac{dS_1}{dt} + (1 - \frac{S_2^*}{S_2})\frac{dS_2}{dt} + (1 - \frac{I_1^*}{I_1})\frac{dI_1}{dt} + \gamma_1(1 - \frac{I_2^*}{I_2})\frac{dI_2}{dt} + \gamma_2(1 - \frac{P^*}{P})\frac{dP}{dt} + \gamma_3(1 - \frac{S_p^*}{S_p})\frac{dS_p}{dt}$
$+\gamma_4(1 - \frac{A^*}{A})\frac{dA}{dt}$

Substituting the expressions for the derivatives in $\frac{dV}{dt}$, it follows that

$\frac{dV}{dt} = (1-\frac{S_1^*}{S_1})[Q_1-[(\frac{\beta_1+\sigma}{N})I_1+(\frac{\beta_2+\sigma}{N})I_2+(\frac{\beta_3+\sigma}{N})P]S_1-\mu S_1]+(1-\frac{S_2^*}{S_2})[Q_2-\frac{\sigma}{N}[I_1+I_2+P]S_2-\mu S_2]+$
$(1-\frac{I_1^*}{I_1})[[(\frac{\beta_1+\sigma}{N})I_1+(\frac{\beta_2+\sigma}{N})I_2+(\frac{\beta_3+\sigma}{N})P]S_1+\frac{\sigma}{N}[I_1+I_2+P]S_2-\nabla_1 I_1]+\gamma_1(1-\frac{I_2^*}{I_2})[\theta I_1-\nabla_2 I_2]+\gamma_2(1-$
$\frac{P^*}{P})[u_1 I_1+u_2 I_2-\nabla_3 P]+\gamma_3(1-\frac{S_p^*}{S_p})[k_1 I_1+k_2 I_2+k_3 P-\mu S_p]+\gamma_4(1-\frac{A^*}{A})[\delta_1 I_1+\delta_2 I_2+\delta_3 P-(\alpha+\mu)A]$

Using the relation $Q_1 = [(\frac{\beta_1+\sigma}{N})I_1^* + (\frac{\beta_2+\sigma}{N})I_2^* + (\frac{\beta_3+\sigma}{N})P^*]S_1^* + \mu S_1^*$, and

$Q_2 = \frac{\sigma}{N}[I_1^* + I_2^* + P^*]S_2^* + \mu S_2^*$ from the first and second equations of the system (5.1)-(5.7) at the steady state then $\frac{dV}{dt}$ can be written as

$$\begin{aligned}\frac{dV}{dt} = &\left(1 - \frac{S_1^*}{S_1}\right)\left[\left[\left(\frac{\beta_1+\sigma}{N^*}\right)I_1^* + \left(\frac{\beta_2+\sigma}{N^*}\right)I_2^* + \left(\frac{\beta_3+\sigma}{N^*}\right)P^*\right]S_1^* + \mu S_1^*\right] - \left[\left(\frac{\beta_1+\sigma}{N}\right)I_1 + \left(\frac{\beta_2+\sigma}{N}\right)I_2\right]S_1\\
&-\left(1 - \frac{S_1^*}{S_1}\right)\left[\left(\frac{\beta_3+\sigma}{N}\right)PS_1 - \mu S_1\right] + \left(1 - \frac{S_2^*}{S_2}\right)\left[\frac{\sigma}{N^*}\left[I_1^* + I_2^* + P^*\right]S_2^* + \mu S_2^* - \frac{\sigma}{N}\left[I_1 + I_2 + P\right]S_2\right]\\
&-\left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N}\mu S_2 + \left(1 - \frac{I_1^*}{I_1}\right)\left[\left[\left(\frac{\beta_1+\sigma}{N}\right)I_1 + \left(\frac{\beta_2+\sigma}{N}\right)I_2 + \left(\frac{\beta_3+\sigma}{N}\right)P\right]S_1 + \frac{\sigma}{N}\left[I_1 + I_2 + P\right]S_2\right]\\
&-\left(1 - \frac{I_1^*}{I_1}\right)\frac{\sigma}{N}\nabla_1 I_1 + \gamma_1\left(1 - \frac{I_2^*}{I_2}\right)\left[\theta I_1 - \nabla_2 I_2\right] + \gamma_2\left(1 - \frac{P^*}{P}\right)\left[u_1 I_1 + u_2 I_2 - \nabla_3 P\right]\\
&+\gamma_3\left(1 - \frac{S_p^*}{S_p}\right)\left[k_1 I_1 + k_2 I_2 + k_3 P - \mu S_p\right] + \gamma_4\left(1 - \frac{A^*}{A}\right)\left[\delta_1 I_1 + \delta_2 I_2 + \delta_3 P - (\alpha+\mu)A\right]\end{aligned}$$

This can then be simplified to

$$\begin{aligned}\frac{dV}{dt} = &\left(1 - \frac{S_1^*}{S_1}\right)\left(\frac{\beta_1+\sigma}{N^*}\right)I_1^* S_1^* + \left(1 - \frac{S_1^*}{S_1}\right)\left(\frac{\beta_2+\sigma}{N^*}\right)I_2^* S_1^* + \left(1 - \frac{S_1^*}{S_1}\right)\left(\frac{\beta_3+\sigma}{N^*}\right)P^* S_1^* +\\
&\mu\left(1 - \frac{S_1^*}{S_1}\right)(S_1^* - S_1) - \left(1 - \frac{S_1^*}{S_1}\right)\left(\frac{\beta_1+\sigma}{N}\right)I_1 S_1 - \left(1 - \frac{S_1^*}{S_1}\right)\left(\frac{\beta_2+\sigma}{N}\right)I_2 S_1 - \left(1 - \frac{S_1^*}{S_1}\right)\left(\frac{\beta_3+\sigma}{N}\right)PS_1\\
&+\left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N^*}I_1^* S_2^* + \left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N^*}I_2^* S_2^* + \left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N^*}P^* S_2^* + \mu\left(1 - \frac{S_2^*}{S_2}\right)(S_2^* - S_2)\\
&-\left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N}I_1 S_2 - \left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N}I_2 S_2 - \left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N}PS_2\\
&+\left(1 - \frac{I_1^*}{I_1}\right)\left[\left[\left(\frac{\beta_1+\sigma}{N}\right)I_1 + \left(\frac{\beta_2+\sigma}{N}\right)I_2 + \left(\frac{\beta_3+\sigma}{N}\right)P\right]S_1 + \frac{\sigma}{N}\left[I_1 + I_2 + P\right]S_2 - \nabla_1 I_1\right]\\
&+\gamma_1\left(1 - \frac{I_2^*}{I_2}\right)\left[\theta I_1 - \nabla_2 I_2\right] + \gamma_2\left(1 - \frac{P^*}{P}\right)\left[u_1 I_1 + u_2 I_2 - \nabla_3 P\right]\\
&+\gamma_3\left(1 - \frac{S_p^*}{S_p}\right)\left[k_1 I_1 + k_2 I_2 + k_3 P - \mu S_p\right] + \gamma_4\left(1 - \frac{A^*}{A}\right)\left[\delta_1 I_1 + \delta_2 I_2 + \delta_3 P - (\alpha+\mu)A\right]\end{aligned}$$

Using the relation at the steady state

$\nabla_1 I_1^* = \left[\left(\frac{\beta_1+\sigma}{N^*}\right)I_1^* + \left(\frac{\beta_2+\sigma}{N^*}\right)I_2^* + \left(\frac{\beta_3+\sigma}{N^*}\right)P^*\right]S_1^* + \frac{\sigma}{N^*}\left[I_1^* + I_2^* + P^*\right]S_2^*,\ \ \nabla_2 I_2^* = \theta I_1^*$

$\nabla_3 P^* = u_1 I_1^* + u_2 I_2^*,\ \mu S_p^* = k_1 I_1^* + k_2 I_2^* + k_3 P^*,\ (\alpha+\mu)A^* = \delta_1 I_1^* + \delta_2 I_2^* + \delta_3 P^*$

We again simplify

$$\begin{aligned}\frac{dV}{dt} = &\left(1 - \frac{S_1^*}{S_1}\right)\left(\frac{\beta_1+\sigma}{N^*}\right)I_1^* S_1^* + \left(1 - \frac{S_1^*}{S_1}\right)\left(\frac{\beta_2+\sigma}{N^*}\right)I_2^* S_1^* + \left(1 - \frac{S_1^*}{S_1}\right)\left(\frac{\beta_3+\sigma}{N^*}\right)P^* S_1^* +\\
&\mu S_1^*\left(2 - \frac{S_1}{S_1^*} - \frac{S_1^*}{S_1}\right) - \left(\frac{\beta_1+\sigma}{N}\right)I_1 S_1 + \left(\frac{\beta_1+\sigma}{N}\right)I_1 S_1^* - \left(\frac{\beta_2+\sigma}{N}\right)I_2 S_1 + \left(\frac{\beta_2+\sigma}{N}\right)I_2 S_1^* - \left(\frac{\beta_3+\sigma}{N}\right)PS_1 +\\
&\left(\frac{\beta_3+\sigma}{N}\right)PS_1^* + \left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N^*}I_1^* S_2^* + \left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N^*}I_2^* S_2^* + \left(1 - \frac{S_2^*}{S_2}\right)\frac{\sigma}{N^*}P^* S_2^* + \mu S_2^*\left(2 - \frac{S_2}{S_2^*} - \frac{S_2^*}{S_2}\right)\\
&-\frac{\sigma}{N}I_1 S_2 + \frac{\sigma}{N}I_1 S_2^* - \frac{\sigma}{N}I_2 S_2 + \frac{\sigma}{N}I_2 S_2^* - \frac{\sigma}{N}PS_2 + \frac{\sigma}{N}PS_2^* + \left(1 - \frac{I_1^*}{I_1}\right)\left(\frac{\beta_1+\sigma}{N}\right)I_1 S_1 +\end{aligned}$$

$$
\begin{aligned}
&\left(1-\tfrac{I_1^*}{I_1}\right)\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1+\left(1-\tfrac{I_1^*}{I_1}\right)\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1+\left(1-\tfrac{I_1^*}{I_1}\right)\tfrac{\sigma}{N}I_1S_2+\left(1-\tfrac{I_1^*}{I_1}\right)\tfrac{\sigma}{N}I_2S_2+\\
&\left(1-\tfrac{I_1^*}{I_1}\right)\tfrac{\sigma}{N}PS_2-\left(1-\tfrac{I_1^*}{I_1}\right)\nabla_1I_1+\gamma_1\left(1-\tfrac{I_2^*}{I_2}\right)[\theta I_1-\nabla_2I_2]+\gamma_2\left(1-\tfrac{P^*}{P}\right)[u_1I_1+u_2I_2-\nabla_3P]\\
&+\gamma_3\left(1-\tfrac{S_p^*}{S_p}\right)[k_1I_1+k_2I_2+k_3P-\mu S_p]+\gamma_4\left(1-\tfrac{A^*}{A}\right)[\delta_1I_1+\delta_2I_2+\delta_3P-(\alpha+\mu)A]\\
=\;&\mu S_1^*\left(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1}\right)+\mu S_2^*\left(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2}\right)+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1^*+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1^*+\\
&\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1^*-\left(\tfrac{\beta_1+\sigma}{N}\right)I_1S_1+\left(\tfrac{\beta_1+\sigma}{N}\right)I_1S_1^*-\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1+\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1^*-\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1\\
&+\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1^*+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}I_1^*S_2^*+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}I_2^*S_2^*+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}P^*S_2^*\\
&-\tfrac{\sigma}{N}I_1S_2+\tfrac{\sigma}{N}I_1S_2^*-\tfrac{\sigma}{N}I_2S_2+\tfrac{\sigma}{N}I_2S_2^*-\tfrac{\sigma}{N}PS_2+\tfrac{\sigma}{N}PS_2^*+\left(\tfrac{\beta_1+\sigma}{N}\right)I_1S_1-\left(\tfrac{\beta_1+\sigma}{N}\right)I_1^*S_1+\\
&\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1-\left(\tfrac{\beta_2+\sigma}{N}\right)\tfrac{I_1^*}{I_1}+\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1-\left(\tfrac{\beta_3+\sigma}{N}\right)\tfrac{I_1^*}{I_1}PS_1+\tfrac{\sigma}{N}I_1S_2-\tfrac{\sigma}{N}I_1^*S_2+\tfrac{\sigma}{N}I_2S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}I_2S_2\\
&+\tfrac{\sigma}{N}PS_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}PS_2-\left(1-\tfrac{I_1^*}{I_1}\right)\nabla_1I_1+\gamma_1\left(1-\tfrac{I_2^*}{I_2}\right)[\theta I_1-\nabla_2I_2]\\
&+\gamma_2\left(1-\tfrac{P^*}{P}\right)[u_1I_1+u_2I_2-\nabla_3P]+\gamma_3\left(1-\tfrac{S_p^*}{S_p}\right)[k_1I_1+k_2I_2+k_3P-\mu S_p]\\
&+\gamma_4\left(1-\tfrac{A^*}{A}\right)[\delta_1I_1+\delta_2I_2+\delta_3P-(\alpha+\mu)A]\\
=\;&\mu S_1^*\left(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1}\right)+\mu S_2^*\left(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2}\right)+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1^*+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1^*+\\
&\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1^*+\left(\tfrac{\beta_1+\sigma}{N}\right)I_1S_1^*+\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1^*+\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1^*+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}I_1^*S_2^*+\\
&\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}I_2^*S_2^*+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}P^*S_2^*+\tfrac{\sigma}{N}I_1S_2^*+\tfrac{\sigma}{N}I_2S_2^*+\tfrac{\sigma}{N}PS_2^*-\left(\tfrac{\beta_1+\sigma}{N}\right)I_1^*S_1+\\
&-\left(\tfrac{\beta_2+\sigma}{N}\right)\tfrac{I_1^*}{I_1}I_2S_1-\left(\tfrac{\beta_3+\sigma}{N}\right)\tfrac{I_1^*}{I_1}PS_1-\tfrac{\sigma}{N}I_1^*S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}I_2S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}PS_2-\left(1-\tfrac{I_1^*}{I_1}\right)\nabla_1I_1\\
&+\gamma_1\left(1-\tfrac{I_2^*}{I_2}\right)[\theta I_1-\nabla_2I_2]+\gamma_2\left(1-\tfrac{P^*}{P}\right)[u_1I_1+u_2I_2-\nabla_3P]\\
&+\gamma_3\left(1-\tfrac{S_p^*}{S_p}\right)[k_1I_1+k_2I_2+k_3P-\mu S_p]+\gamma_4\left(1-\tfrac{A^*}{A}\right)[\delta_1I_1+\delta_2I_2+\delta_3P-(\alpha+\mu)A]\\
=\;&\mu S_1^*\left(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1}\right)+\mu S_2^*\left(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2}\right)+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1^*+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1^*+\\
&+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1^*+\left(\tfrac{\beta_1+\sigma}{N}\right)I_1S_1^*+\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1^*+\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1^*+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}I_1^*S_2^*\\
&+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}I_2^*S_2^*+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}P^*S_2^*+\tfrac{\sigma}{N}I_1S_2^*+\tfrac{\sigma}{N}I_2S_2^*+\tfrac{\sigma}{N}PS_2^*-\left(\tfrac{\beta_1+\sigma}{N}\right)I_1^*S_1\\
&-\left(\tfrac{\beta_2+\sigma}{N}\right)\tfrac{I_1^*}{I_1}I_2S_1-\left(\tfrac{\beta_3+\sigma}{N}\right)\tfrac{I_1^*}{I_1}PS_1-\tfrac{\sigma}{N}I_1^*S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}I_2S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}PS_2-\nabla_1I_1+\nabla_1I_1^*\\
&+\gamma_1\left(1-\tfrac{I_2^*}{I_2}\right)[\theta I_1-\nabla_2I_2]+\gamma_2\left(1-\tfrac{P^*}{P}\right)[u_1I_1+u_2I_2-\nabla_3P]\\
&+\gamma_3\left(1-\tfrac{S_p^*}{S_p}\right)[k_1I_1+k_2I_2+k_3P-\mu S_p]+\gamma_4\left(1-\tfrac{A^*}{A}\right)[\delta_1I_1+\delta_2I_2+\delta_3P-(\alpha+\mu)A]\\
=\;&\mu S_1^*\left(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1}\right)+\mu S_2^*\left(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2}\right)+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1^*+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1^*\\
&+\left(1-\tfrac{S_1^*}{S_1}\right)\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1^*+\left(\tfrac{\beta_1+\sigma}{N}\right)I_1S_1^*+\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1^*+\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1^*+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}I_1^*S_2^*\\
&+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}I_2^*S_2^*+\left(1-\tfrac{S_2^*}{S_2}\right)\tfrac{\sigma}{N^*}P^*S_2^*+\tfrac{\sigma}{N}I_1S_2^*+\tfrac{\sigma}{N}I_2S_2^*+\tfrac{\sigma}{N}PS_2^*-\left(\tfrac{\beta_1+\sigma}{N}\right)I_1^*S_1\\
&-\left(\tfrac{\beta_2+\sigma}{N}\right)\tfrac{I_1^*}{I_1}I_2S_1-\left(\tfrac{\beta_3+\sigma}{N}\right)\tfrac{I_1^*}{I_1}PS_1-\tfrac{\sigma}{N}I_1^*S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}I_2S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}PS_2+\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1^*+\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1^*\\
&+\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1^*+\tfrac{\sigma}{N^*}I_1^*S_2^*+\tfrac{\sigma}{N^*}I_2^*S_2^*+\tfrac{\sigma}{N^*}P^*S_2^*-\nabla_1I_1+\gamma_1\left(1-\tfrac{I_2^*}{I_2}\right)[\theta I_1-\nabla_2I_2]\\
&+\gamma_2\left(1-\tfrac{P^*}{P}\right)[u_1I_1+u_2I_2-\nabla_3P]+\gamma_3\left(1-\tfrac{S_p^*}{S_p}\right)[k_1I_1+k_2I_2+k_3P-\mu S_p]\\
&+\gamma_4\left(1-\tfrac{A^*}{A}\right)[\delta_1I_1+\delta_2I_2+\delta_3P-(\alpha+\mu)A]\\
=\;&\mu S_1^*\left(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1}\right)+\mu S_2^*\left(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2}\right)+\left(2-\tfrac{S_1^*}{S_1}-\tfrac{S_1}{S_1^*}\right)\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1^*+\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1\\
&+\left(2-\tfrac{S_1^*}{S_1}-\tfrac{S_1}{S_1^*}\right)\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1^*+\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1+\left(2-\tfrac{S_1^*}{S_1}-\tfrac{S_1}{S_1^*}\right)\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1^*+\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1\\
&+\left(\tfrac{\beta_1+\sigma}{N}\right)I_1S_1^*+\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1^*+\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1^*+\left(2-\tfrac{S_2^*}{S_2}-\tfrac{S_2}{S_2^*}\right)\tfrac{\sigma}{N^*}I_1^*S_2^*+\tfrac{\sigma}{N^*}I_1^*S_1\\
&+\left(2-\tfrac{S_2^*}{S_2}-\tfrac{S_2}{S_2^*}\right)\tfrac{\sigma}{N^*}I_2^*S_2^*+\tfrac{\sigma}{N^*}I_2^*S_1+\left(2-\tfrac{S_2^*}{S_2}-\tfrac{S_2}{S_2^*}\right)\tfrac{\sigma}{N^*}P^*S_2^*+\tfrac{\sigma}{N^*}P^*S_2+\tfrac{\sigma}{N}I_1S_2^*+\tfrac{\sigma}{N}I_2S_2^*
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{\sigma}{N}PS_2^*-\left(\tfrac{\beta_1+\sigma}{N}\right)I_1^*S_1-\left(\tfrac{\beta_2+\sigma}{N}\right)\tfrac{I_1^*}{I_1}I_2S_1-\left(\tfrac{\beta_3+\sigma}{N}\right)\tfrac{I_1^*}{I_1}PS_1-\tfrac{\sigma}{N}I_1^*S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}I_2S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}PS_2-\nabla_1 I_1\\
&+\gamma_1\left[\theta I_1+\nabla_2 I_2^*-\theta\tfrac{I_2^*}{I_2}I_1-\nabla_2 I_2\right]+\gamma_2\left[u_1I_1+u_2I_2+\nabla_3P^*-u_1\tfrac{P^*}{P}I_1-u_2\tfrac{P^*}{P}I_2-\nabla_3P\right]\\
&+\gamma_3\left[k_1I_1+k_2I_2+k_3P+\mu S_p^*-k_1\tfrac{S_p^*}{S_p}I_1-k_2\tfrac{S_p^*}{S_p}I_2-k_3\tfrac{S_p^*}{S_p}P-\mu S_p\right]\\
&+\gamma_4\left[\delta_1I_1+\delta_2I_2+\delta_3P+(\alpha+\mu)A^*-\delta_1\tfrac{A^*}{A}I_1+\delta_2\tfrac{A^*}{A}I_2+\delta_3\tfrac{A^*}{A}P-(\alpha+\mu)A\right]\\
=\;&\mu S_1^*\left(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1}\right)+\mu S_2^*\left(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2}\right)+\left(2-\tfrac{S_1^*}{S_1}-\tfrac{S_1}{S_1^*}\right)\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1^*\\
&+\left(2-\tfrac{S_1^*}{S_1}-\tfrac{S_1}{S_1^*}\right)\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1^*+\left(2-\tfrac{S_1^*}{S_1}-\tfrac{S_1}{S_1^*}\right)\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1^*+\left(2-\tfrac{S_2^*}{S_2}-\tfrac{S_2}{S_2^*}\right)\tfrac{\sigma}{N^*}I_1^*S_2^*\\
&+\left(2-\tfrac{S_2^*}{S_2}-\tfrac{S_2}{S_2^*}\right)\tfrac{\sigma}{N^*}I_2^*S_2^*+\left(2-\tfrac{S_2^*}{S_2}-\tfrac{S_2}{S_2^*}\right)\tfrac{\sigma}{N^*}P^*S_2^*+\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1+\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1\\
&+\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1+\left(\tfrac{\beta_1+\sigma}{N}\right)I_1S_1^*+\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1^*+\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1^*+\tfrac{\sigma}{N^*}I_1^*S_1+\tfrac{\sigma}{N^*}I_2^*S_1+\tfrac{\sigma}{N^*}P^*S_2\\
&+\tfrac{\sigma}{N}I_1S_2^*+\tfrac{\sigma}{N}I_2S_2^*+\tfrac{\sigma}{N}PS_2^*-\left(\tfrac{\beta_1+\sigma}{N}\right)I_1^*S_1-\left(\tfrac{\beta_2+\sigma}{N}\right)\tfrac{I_1^*}{I_1}I_2S_1-\left(\tfrac{\beta_3+\sigma}{N}\right)\tfrac{I_1^*}{I_1}PS_1-\tfrac{\sigma}{N}I_1^*S_2\\
&-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}I_2S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}PS_2-\nabla_1I_1+\gamma_1\left(\theta I_1+\nabla_2I_2^*\right)-\gamma_1\left(\theta\tfrac{I_2^*}{I_2}I_1+\nabla_2I_2\right)+\gamma_2\left(u_1I_1+u_2I_2+\nabla_3P^*\right)\\
&-\gamma_2\left(u_1\tfrac{P^*}{P}I_1+u_2\tfrac{P^*}{P}I_2+\nabla_3P\right)+\gamma_3\left(k_1I_1+k_2I_2+k_3P+\mu S_p^*\right)\\
&-\gamma_3\left(k_1\tfrac{S_p^*}{S_p}I_1+k_2\tfrac{S_p^*}{S_p}I_2+k_3\tfrac{S_p^*}{S_p}P+\mu S_p\right)+\gamma_4\left(\delta_1I_1+\delta_2I_2+\delta_3P+(\alpha+\mu)A^*\right)\\
&-\gamma_4\left(\delta_1\tfrac{A^*}{A}I_1+\delta_2\tfrac{A^*}{A}I_2+\delta_3\tfrac{A^*}{A}P+(\alpha+\mu)A\right)
\end{aligned}
$$

The coefficients $\gamma_1,\gamma_2,\gamma_3$,and $\gamma_4$ are obtained from the following system derived from the above expression

$\gamma_1\theta I_1-\nabla_1I_1+\gamma_2u_1I_1+\gamma_3k_1I_1+\gamma_4\delta_1I_1=0$

$-\gamma_1\nabla_2I_2+\gamma_2u_2I_2+\gamma_3k_2I_2+\gamma_4\delta_2I_2=0$

$-\gamma_2\nabla_3P+\gamma_3k_3P+\gamma_4\delta_3P=0$

Let $\gamma_4=0$, then the above equation reduced to

$$\gamma_1\theta I_1-\nabla_1I_1+\gamma_2u_1I_1+\gamma_3k_1I_1=0 \tag{5.20}$$

$$-\gamma_1\nabla_2I_2+\gamma_2u_2I_2+\gamma_3k_2I_2=0 \tag{5.21}$$

$$-\gamma_2\nabla_3P+\gamma_3k_3P=0 \tag{5.22}$$

From (5.22) we have $\gamma_3=\frac{\gamma_2\nabla_3P}{k_3P}=\frac{\gamma_2\nabla_3}{k_3}$ substituting this in to (5.20) and (5.21) we get

$$
\begin{cases}\gamma_1\theta-\nabla_1+\gamma_2u_1+\frac{\gamma_2\nabla_3}{k_3}k_1=0\\ -\gamma_1\nabla_2+\gamma_2u_2+\frac{\gamma_2\nabla_3}{k_3}k_2=0\end{cases}
\Rightarrow\begin{cases}\gamma_1\theta+\gamma_2\left(u_1+\frac{k_1\nabla_3}{k_3}\right)=\nabla_1\\ -\gamma_1\nabla_2+\gamma_2\left(u_2+\frac{k_2\nabla_3}{k_3}\right)=0\end{cases}
$$

From $-\gamma_1\nabla_2+\gamma_2\left(u_2+\frac{k_2\nabla_3}{k_3}\right)=0$ we have $\gamma_1=\gamma_2\left(u_2+\frac{k_2\nabla_3}{k_3}\right)\frac{1}{\nabla_2}$ again putting this in to $\gamma_1\theta+\gamma_2\left(u_1+\frac{k_1\nabla_3}{k_3}\right)=\nabla_1$ we get $\gamma_2\left(u_2+\frac{k_2\nabla_3}{k_3}\right)\frac{1}{\nabla_2}\theta+\gamma_2\left(u_1+\frac{k_1\nabla_3}{k_3}\right)=\nabla_1$ implies

$\gamma_2=\frac{\nabla_1}{\left(u_2+\frac{k_2\nabla_3}{k_3}\right)\frac{1}{\nabla_2}\theta+\left(u_1+\frac{k_1\nabla_3}{k_3}\right)}$ and hence

$$\gamma_1 = \frac{\nabla_1\left(u_2+\frac{k_2\nabla_3}{k_3}\right)\frac{1}{\nabla_2}}{\left(u_2+\frac{k_2\nabla_3}{k_3}\right)\frac{1}{\nabla_2}\theta+\left(u_1+\frac{k_1\nabla_3}{k_3}\right)} = \frac{\nabla_1\left(u_2+\frac{k_2\nabla_3}{k_3}\right)}{\left(u_2+\frac{k_2\nabla_3}{k_3}\right)\theta+\nabla_2\left(u_1+\frac{k_1\nabla_3}{k_3}\right)}$$

$$\gamma_3 = \frac{\nabla_1\nabla_3}{k_3\left(\left(u_2+\frac{k_2\nabla_3}{k_3}\right)\frac{1}{\nabla_2}\theta+\left(u_1+\frac{k_1\nabla_3}{k_3}\right)\right)}$$

Thus

$$\begin{aligned}\frac{dV}{dt} &= \mu S_1^*\left(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1}\right)+\mu S_2^*\left(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2}\right)+\left(2-\tfrac{S_1^*}{S_1}-\tfrac{S_1}{S_1^*}\right)\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1^*\\
&+\left(2-\tfrac{S_1^*}{S_1}-\tfrac{S_1}{S_1^*}\right)\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1^*+\left(2-\tfrac{S_1^*}{S_1}-\tfrac{S_1}{S_1^*}\right)\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1^*+\left(2-\tfrac{S_2^*}{S_2}-\tfrac{S_2}{S_2^*}\right)\tfrac{\sigma}{N^*}I_1^*S_2^*\\
&+\left(2-\tfrac{S_2^*}{S_2}-\tfrac{S_2}{S_2^*}\right)\tfrac{\sigma}{N^*}I_2^*S_2^*+\left(2-\tfrac{S_2^*}{S_2}-\tfrac{S_2}{S_2^*}\right)\tfrac{\sigma}{N^*}P^*S_2^*+\left(\tfrac{\beta_1+\sigma}{N^*}\right)I_1^*S_1+\left(\tfrac{\beta_2+\sigma}{N^*}\right)I_2^*S_1+\left(\tfrac{\beta_3+\sigma}{N^*}\right)P^*S_1\\
&+\left(\tfrac{\beta_1+\sigma}{N}\right)I_1S_1^*+\left(\tfrac{\beta_2+\sigma}{N}\right)I_2S_1^*+\left(\tfrac{\beta_3+\sigma}{N}\right)PS_1^*+\tfrac{\sigma}{N^*}I_1^*S_1+\tfrac{\sigma}{N^*}I_2^*S_1+\tfrac{\sigma}{N^*}P^*S_2\\
&+\tfrac{\sigma}{N}I_1S_2^*+\tfrac{\sigma}{N}I_2S_2^*+\tfrac{\sigma}{N}PS_2^*-\left(\tfrac{\beta_1+\sigma}{N}\right)I_1^*S_1-\left(\tfrac{\beta_2+\sigma}{N}\right)\tfrac{I_1^*}{I_1}I_2S_1-\left(\tfrac{\beta_3+\sigma}{N}\right)\tfrac{I_1^*}{I_1}PS_1-\tfrac{\sigma}{N}I_1^*S_2\\
&-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}I_2S_2-\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}PS_2-\nabla_1I_1+\gamma_1\left(\theta I_1+\nabla_2I_2^*\right)-\gamma_1\left(\theta\tfrac{I_2^*}{I_2}I_1+\nabla_2I_2\right)\\
&+\gamma_2\left(u_1I_1+u_2I_2+\nabla_3P^*\right)-\gamma_2\left(u_1\tfrac{P^*}{P}I_1+u_2\tfrac{P^*}{P}I_2+\nabla_3P\right)\\
&+\gamma_3\left(k_1I_1+k_2I_2+k_3P+\mu S_p^*\right)-\gamma_3\left(k_1\tfrac{S_p^*}{S_p}I_1+k_2\tfrac{S_p^*}{S_p}I_2+k_3\tfrac{S_p^*}{S_p}P+\mu S_p\right)\end{aligned}$$

Note that $(2-\frac{S_1}{S_1^*}-\frac{S_1^*}{S_1})$ and $(2-\frac{S_2}{S_2^*}-\frac{S_2^*}{S_2})$ are less than or equal to zero by arithmetic mean -geometric mean inequality.

This gives

$$\frac{dV}{dt} = Z - Y$$

Where

$$\begin{aligned}Z = \gamma_1(\theta I_1+\nabla_2I_2^*)+\gamma_2(u_1I_1+u_2I_2+\nabla_3P^*)+\gamma_3(k_1I_1+k_2I_2+k_3P+\mu S_p^*)+(\frac{\beta_1+\sigma}{N^*})I_1^*S_1\\
+(\frac{\beta_2+\sigma}{N^*})I_2^*S_1+(\frac{\beta_3+\sigma}{N^*})P^*S_1+(\frac{\beta_1+\sigma}{N})I_1S_1^*+(\frac{\beta_2+\sigma}{N})I_2S_1^*+(\frac{\beta_3+\sigma}{N})PS_1^*+\frac{\sigma}{N^*}I_1^*S_1+\\
\frac{\sigma}{N^*}I_2^*S_1+\frac{\sigma}{N^*}P^*S_2+\frac{\sigma}{N}I_1S_2^*+\frac{\sigma}{N}I_2S_2^*+\frac{\sigma}{N}PS_2^*\end{aligned}$$

and

$$\begin{aligned}Y = -[&\mu S_1^*(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1})+\mu S_2^*(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2})+(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1})(\tfrac{\beta_1+\sigma}{N^*})I_1^*S_1^*\\
&+(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1})(\tfrac{\beta_2+\sigma}{N^*})I_2^*S_1^*+(2-\tfrac{S_1}{S_1^*}-\tfrac{S_1^*}{S_1})(\tfrac{\beta_3+\sigma}{N^*})P^*S_1^*\\
&+(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2})\tfrac{\sigma}{N^*}I_1^*S_2^*+(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2})\tfrac{\sigma}{N^*}I_2^*S_2^*+(2-\tfrac{S_2}{S_2^*}-\tfrac{S_2^*}{S_2})\tfrac{\sigma}{N^*}P^*S_2^*]\\
&+[(\tfrac{\beta_1+\sigma}{N})I_1^*S_1+(\tfrac{\beta_2+\sigma}{N})\tfrac{I_1^*}{I_1}I_2S_1+(\tfrac{\beta_3+\sigma}{N})\tfrac{I_1^*}{I_1}PS_1+\tfrac{\sigma}{N}I_1^*S_2+\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}I_2S_2\\
&+\tfrac{\sigma}{N}\tfrac{I_1^*}{I_1}PS_2+\nabla_1I_1+\gamma_1(\theta\tfrac{I_2^*}{I_2}I_1+\nabla_2I_2)+\gamma_2(u_1\tfrac{P^*}{P}I_1+u_2\tfrac{P^*}{P}I_2+\nabla_3P)\\
&+\gamma_3(k_1\tfrac{S_p^*}{S_p}I_1+k_2\tfrac{S_p^*}{S_p}I_2+k_3\tfrac{S_p^*}{S_p}P+\mu S_p)]\end{aligned}$$

Hence, if $Z < Y$ then, $\frac{dV}{dt}$ will be negative definite, implying that $\frac{dV}{dt} < 0$. Also $\frac{dV}{dt} = 0$ if and only if $S_1 = S_1^*, S_2 = S_2^*, I_1 = I_1^*, I_2 = I_2^*, P = P^*, S_p = S_p^*$ and $A = A^*$.

Therefore, the largest compact invariant set in $\{(S_1^*, S_2^*, I_1^*, I_2^*, P^*, S_p^*, A^*) \in \Omega : \frac{dV}{dt} = 0\}$ is

the singleton $\{E^*\}$, where $E^*$ is endemic equilibrium of the system (5.1)-(5.7). By LaSalle's invariant principle, it then implies that $E^*$ is globally asymptotically stable in $\Omega$ if $Z < Y$. $\square$

# Chapter 6

# Sensitivity analysis and Numerical simulation of Treatment and Inflow Infective Immigrants on the Dynamics of HIV/AIDS

This chapter will discuss about Sensitivity analysis and Numerical simulation of the model stated under chapter four.

The parameter values and assumptions of any model are subject to change and error. Sensitivity analysis is the investigation of these potential changes & errors and their impacts on conclusions to be drawn from the model. Here we use it to discover parameters that have a high impact on reproduction number $R_0$. We calculate the sensitivity indices of the reproductive number $R_0$, to the parameters in the model. These indices tell us how crucial each parameter is to disease transmission and prevalence. The normalized forward sensitivity index of a variable to a parameter is the ratio of the relative change in the variable to the relative change in the parameter. When the variable is a differentiable function of the parameter, the sensitivity index may be alternatively defined using partial derivatives [78].

**Definition**: The normalized forward sensitivity index of a variable, $u$ that depends differentiable on a parameter $p$, is defined as:

$$SI(p) = \frac{\partial u}{\partial p} \times \frac{p}{u}$$

If the magnitude of sensitivity index is high for the parameter $p$ out of other parameters then we say that $p$ is more sensitive parameter.

Here we consider parameters $\beta_1, \beta_2, \sigma, \delta_1, \delta_2, \theta, k_1, k_2, \mu, p_1, p_2$ and $\phi$ to see the sensitivity parameter with regard to basic reproduction number $R_0$ as follows:

$$SI\left(\beta_1\right) = \frac{\partial R_0}{\partial \beta_1} x \frac{\beta_1}{R_0} = \frac{\beta_1\left(k_2+\delta_2+\mu-p_2\right)}{\left(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi\right)\left(k_2+\delta_2+\mu-p_2\right)R_0}$$

$$= \frac{\beta_1\left(k_2+\delta_2+\mu-p_2\right)}{\Delta_1\Delta_2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{\beta_1\left(k_2+\delta_2+\mu-p_2\right)}{\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)}$$

$$SI\left(\beta_2\right) = \frac{\partial R_0}{\partial \beta_2} x \frac{\beta_2}{R_0} = \frac{\theta\beta_2}{\left(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi\right)\left(k_2+\delta_2+\mu-p_2\right)R_0}$$

$$= \frac{\theta\beta_2}{\Delta_1\Delta_2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{\theta\beta_2}{\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)}$$

$$SI\left(\sigma\right) = \frac{\partial R_0}{\partial \sigma} x \frac{\sigma}{R_0} = \frac{\sigma\left(k_2+\delta_2+\mu+\theta-p_2\right)}{\left(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi\right)\left(k_2+\delta_2+\mu-p_2\right)R_0}$$

$$= \frac{\sigma\left(k_2+\delta_2+\mu+\theta-p_2\right)}{\Delta_1\Delta_2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{\sigma\left(k_2+\delta_2+\mu+\theta-p_2\right)}{\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)}$$

$$SI\left(\delta_1\right) = \frac{\partial R_0}{\partial \delta_1} x \frac{\delta_1}{R_0} = \frac{-\delta_1\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi\right)^2\left(k_2+\delta_2+\mu-p_2\right)R_0}$$

$$= \frac{-\delta_1\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{{\Delta_1}^2\Delta_2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{-\delta_1\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$$= \frac{-\delta_1}{\left(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi\right)}$$

$$SI\left(\delta_2\right) = \frac{\partial R_0}{\partial \delta_2} x \frac{\delta_2}{R_0} = \frac{\delta_2\left[\left(\beta_1+\sigma\right)\left(k_2+\delta_2+\mu-p_2\right)-\left(\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi\right)\left(k_2+\delta_2+\mu-p_2\right)^2 R_0}$$

$$= \frac{\delta_2\left[\left(\beta_1+\sigma\right)\left(k_2+\delta_2+\mu-p_2\right)-\left(\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right)\right]}{\Delta_1{\Delta_2}^2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{-\delta_2\theta\left(\beta_2+\sigma\right)}{\left(k_2+\delta_2+\mu-p_2\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$$SI\left(k_1\right) = \frac{\partial R_0}{\partial k_1} x \frac{k_1}{R_0} = \frac{-k_1\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi\right)^2\left(k_2+\delta_2+\mu-p_2\right)R_0}$$

$$= \frac{-k_1\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{{\Delta_1}^2\Delta_2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{-k_1\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$$= \frac{-k_1}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)}$$

$SI\left(k_2\right) = \frac{\partial R_0}{\partial k_2} x \frac{k_2}{R_0}$

$$= \frac{k_2\left[\left(\beta_1+\sigma\right)\left(k_2+\delta_2+\mu-p_2\right)-\left(\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)\left(k_2+\delta_2+\mu-p_2\right)^2 R_0}$$

$$= \frac{k_2\left[\left(\beta_1+\sigma\right)\left(k_2+\delta_2+\mu-p_2\right)-\left(\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right)\right]}{\Delta_1{\Delta_2}^2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{-k_2\theta\left(\beta_2+\sigma\right)}{\left(k_2+\delta_2+\mu-p_2\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$SI\left(p_1\right) = \frac{\partial R_0}{\partial p_1} x \frac{p_1}{R_0} = \frac{p_1[(k_2+\delta_2+\mu-p_2)(\beta_1+\sigma)+\theta(\beta_2+\sigma)]}{(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\phi)^2(k_2+\delta_2+\mu-p_2)R_0}$

$$= \frac{p_1\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{{\Delta_1}^2\Delta_2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{p_1\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$$= \frac{p_1}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)}$$

$SI\left(p_2\right) = \frac{\partial R_0}{\partial p_2} x \frac{p_2}{R_0} = \frac{p_2[-(\beta_1+\sigma)(k_2+\delta_2+\mu-p_2)-((k_2+\delta_2+\mu-p_2)(\beta_1+\sigma)+\theta(\beta_2+\sigma))(-1)]}{(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\phi)(k_2+\delta_2+\mu-p_2)^2 R_0}$

$$= \frac{p_2\left[\theta\left(\beta_2+\sigma\right)\right]}{\Delta_1{\Delta_2}^2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{p_2\left[\theta\left(\beta_2+\sigma\right)\right]}{\left(k_2+\delta_2+\mu-p_2\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$SI\left(\theta\right) = \frac{\partial R_0}{\partial \theta} x \frac{\theta}{R_0}$

$= \frac{\theta[(\beta_2+\sigma)(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\phi)-((k_2+\delta_2+\mu-p_2)(\beta_1+\sigma)+\theta(\beta_2+\sigma))]}{(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\phi)^2(k_2+\delta_2+\mu-p_2)R_0}$

$$= \frac{\theta\left[\left(k_1+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)-\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)\right]}{{\Delta_1}^2\Delta_2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{\theta\left[\left(k_1+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)-\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$SI\left(\phi\right) = \frac{\partial R_0}{\partial \phi} x \frac{\phi}{R_0} = \frac{\phi(1-\epsilon)[(k_2+\delta_2+\mu-p_2)(\beta_1+\sigma)+\theta(\beta_2+\sigma)]}{(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\phi)^2(k_2+\delta_2+\mu-p_2)R_0}$

$$= \frac{\phi\left(1-\epsilon\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{{\Delta_1}^2\Delta_2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{\phi\left(1-\epsilon\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$$= \frac{\phi\left(1-\epsilon\right)}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)}$$

$SI\left(\mu\right) = \frac{\partial R_0}{\partial \mu} x \frac{\mu}{R_0}$

$= \frac{\mu[(\beta_1+\sigma)\Delta_1\Delta_2-(\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma))[(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\phi)+(k_2+\delta_2+\mu-p_2)]}{(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\phi)^2(k_2+\delta_2+\mu-p_2)^2 R_0}$

$$= \frac{\mu\left[\left(\beta_1+\sigma\right)\Delta_1\Delta_2-\left(\Delta_2\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right)[\ \Delta_1+\Delta_2]\right]}{{\Delta_1}^2{\Delta_2}^2\left(\frac{\Delta_2(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{\Delta_1\Delta_2}\right)}$$

$$= \frac{-\mu\left[\Delta_1\theta\left(\beta_2+\sigma\right)+{\Delta_2}^2\left(\beta_1+\sigma\right)+\Delta_2\theta\left(\beta_1+\sigma\right)\right]}{\Delta_1\Delta_2\left[\Delta_2\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$$= -\mu\left[\frac{\Delta_1\theta\left(\beta_2+\sigma\right)}{\Delta_1\Delta_2\left[\Delta_2\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}+\frac{{\Delta_2}^2\left(\beta_1+\sigma\right)+\Delta_2\theta\left(\beta_1+\sigma\right)}{\Delta_1\Delta_2\left[\Delta_2\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}\right]$$

$$= -\mu\left[\frac{\theta\left(\beta_2+\sigma\right)}{\left(k_2+\delta_2+\mu-p_2\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}\right]$$

$$-\mu\left[\frac{1}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)}\right]$$

## 6.1 Parameter Values for Numerical Simulation and Sensitivity Analysis

To perform numerical simulation and sensitivity analysis we collected the following parameter values from different data sources.

Table 6.1: Description of parameters and parameter values

| **Parameter** | **Parameter Description** | **Value** | **Ref.** |
|---|---|---|---|
| $\beta_1$ | Probability of transmission of the disease to susceptible individuals by unaware infective | 0.9 | [104] |
| $\beta_2$ | Probability of transmission of the disease to susceptible individuals by aware infective | 0.7 | [104] |
| $\delta_1$ | Rate of development to AIDS from unaware infective | 0.3 | [104] |
| $\delta_2$ | Rate of development to AIDS from aware infective | 0.02 | [104] |
| $\sigma$ | Rate of transmission through blood borne | 0.003 | [50] |
| $\mu$ | Natural mortality | 0.02 | [60] |

| | | | |
|---|---|---|---|
| $\theta$ | Rate of status awareness due to screening method | 0.3 | [104] |
| $k_1$ | Rate of treatment of unaware infective | 0.1 | Assumed |
| $k_2$ | Rate of treatment of aware infective | 0.4 | [79] |
| $p_1$ | Rate of unaware infective immigrants | 0.1 | [104] |
| $p_2$ | Rate of aware infective immigrants | 0.2 | [104] |
| $\phi$ | Rate of vertical transmission | 0.03 | [104] |
| $\epsilon$ | Probability of death at birth | 0.2 | [104] |
| $\alpha$ | AIDS induced death rate | 0.9 | [104] |

We calculate the reproduction number $R_0$ using values shown in table 6.1

$$R_0 = \frac{(k_2+\delta_2+\mu-p_2)(\beta_1+\sigma)+\theta(\beta_2+\sigma)}{(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi)(k_2+\delta_2+\mu-p_2)}$$

$$= \frac{(0.4+0.02+0.02-0.2)\,(0.9+0.003)+0.3\,(0.7+0.003)}{(0.1+0.3+0.3+0.02-0.1-\ (1-0.2)\,0.03)\,(0.4+0.02+0.02-0.2)}$$

$$= \frac{(0.24)\,(0.903)+0.3\,(0.703)}{(0.62-\ 0.024)\,0.24} = \frac{0.21672+0.2109}{(0.596)\,0.24}$$

$$= \frac{0.42762}{(0.596)\,(0.24)} = \frac{0.42762}{0.14304} = 2.9895$$

$$\Rightarrow R_0 = 2.99$$

This tells us that the disease persists in the population.

We calculate the sensitivity indices of $R_0$ using the derived formula above for each model parameter using values shown in table 6.1. These are:

$$SI\,(\beta_1) = \frac{\beta_1\,(k_2+\delta_2+\mu-p_2)}{(k_2+\delta_2+\mu-p_2)\,(\beta_1+\sigma)+\theta\,(\beta_2+\sigma)} = 0.51$$

$$SI\,(\beta_2) = \frac{\theta\beta_2}{(k_2+\delta_2+\mu-p_2)\,(\beta_1+\sigma)+\theta\,(\beta_2+\sigma)} = 0.49$$

$$SI\,(\sigma) = \frac{\sigma\,(k_2+\delta_2+\mu+\theta-p_2)}{(k_2+\delta_2+\mu-p_2)\,(\beta_1+\sigma)+\theta\,(\beta_2+\sigma)} = 0.00$$

$$SI\,(\delta_1) = \frac{-\delta_1}{(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi)} = -0.50$$

$$SI\,(\delta_2) = \frac{-\delta_2\theta\,(\beta_2+\sigma)}{(k_2+\delta_2+\mu-p_2)\,[(k_2+\delta_2+\mu-p_2)\,(\beta_1+\sigma)+\theta\,(\beta_2+\sigma)]} = -0.04$$

$$SI\,(k_1) = \frac{-k_1}{(k_1+\theta+\delta_1+\mu-p_1-\ (1-\epsilon)\,\phi)} = -0.17$$

$$SI\,(k_2) = \frac{-k_2\theta\,(\beta_2+\sigma)}{(k_2+\delta_2+\mu-p_2)\,[(k_2+\delta_2+\mu-p_2)\,(\beta_1+\sigma)+\theta\,(\beta_2+\sigma)]} = -0.28$$

$$SI\left(p_1\right)=\frac{p_1}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)}=0.17$$

$$SI\left(p_2\right)=\frac{p_2\left[\theta\left(\beta_2+\sigma\right)\right]}{\left(k_2+\delta_2+\mu-p_2\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}=0.41$$

$$SI\left(\theta\right)=\frac{\theta\left[\left(k_1+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)-\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)\right]}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$$=0.09$$

$$SI\left(\phi\right)=\frac{\phi\left(1-\epsilon\right)}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)}=0.04$$

$$SI\left(\mu\right)=-\mu[\frac{\theta\left(\beta_2+\sigma\right)}{\left(k_2+\delta_2+\mu-p_2\right)\left[\left(k_2+\delta_2+\mu-p_2\right)\left(\beta_1+\sigma\right)+\theta\left(\beta_2+\sigma\right)\right]}$$

$$+\frac{1}{\left(k_1+\theta+\delta_1+\mu-p_1-\ \left(1-\epsilon\right)\phi\right)}]=-0.07$$

The results are displayed in Table 6.2.

Table 6.2: Sensitivity indices of $R_0$

| **Parameter** | **Sensitivity Index** |
|---|---|
| $\beta_1$ | 0.51 |
| $\delta_1$ | -0.50 |
| $\beta_2$ | 0.49 |
| $p_2$ | 0.41 |
| $k_2$ | -0.28 |
| $p_1$ | 0.17 |
| $k_1$ | -0.17 |
| $\theta$ | 0.09 |
| $\mu$ | -0.07 |
| $\delta_2$ | -0.04 |
| $\phi$ | 0.04 |
| $\sigma$ | 0.00 |

Table 6.2 contains positive and negative sensitivity indices. The parameters are ordered from most sensitive to least. The most sensitive parameter is the probability of the disease transmits to susceptible people by unaware infective humans, $\beta_1$ and the least sensitive parameter is the rate of transmission through blood borne, $\sigma$. The indices having positive signs increase the value of $R_0$ as one increase them and those having negative signs decrease the value of $R_0$,

when they are increased.

One of the most sensitive parameter is $\beta_1$. Increasing (or decreasing) $\beta_1$ by 10%, increases (or decreases) $R_0$ by 5.1%. Also, since the sensitivity index of $\delta_1 = -0.50$ , increasing $\delta_1$ by 10%, decreases $R_0$ by 5.0%. We can similarly interpret the remaining parameter values. The effectiveness of varying each parameter can thus be determined. As regards HIV/AIDS control, interventions must target the most sensitive parameters. For instance, interventions that target person to person transmission of the disease are the most effective in controlling the disease.

## 6.2 Numerical Simulation

In this section we will discuss the relationship between basic reproduction number and a parameter using graphs as follows.

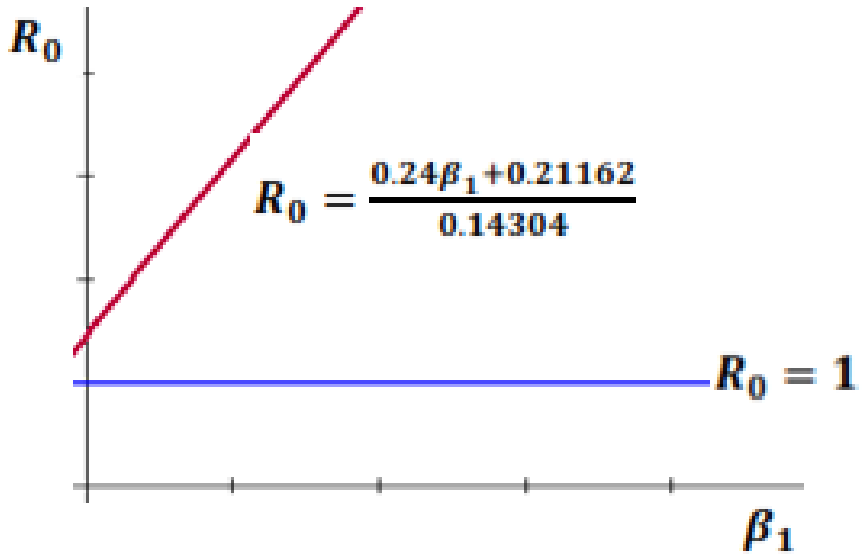


Figure 6.1: the relationship between the reproduction number and the rate of horizontal transmission of the disease from unaware infective class.

From figure 6.1, we can observe that an increase in the rate of transmission, $\beta_1$, makes an increase in the reproduction number, $R_0$. That is the disease always persists for any value of parameter $\beta_1$.

From figure 6.2, we can observe that an increase in the rate of transmission, $\beta_2$, makes an increase in the reproduction number, $R_0$ . That is the disease always persists for any value of parameter $\beta_2$.

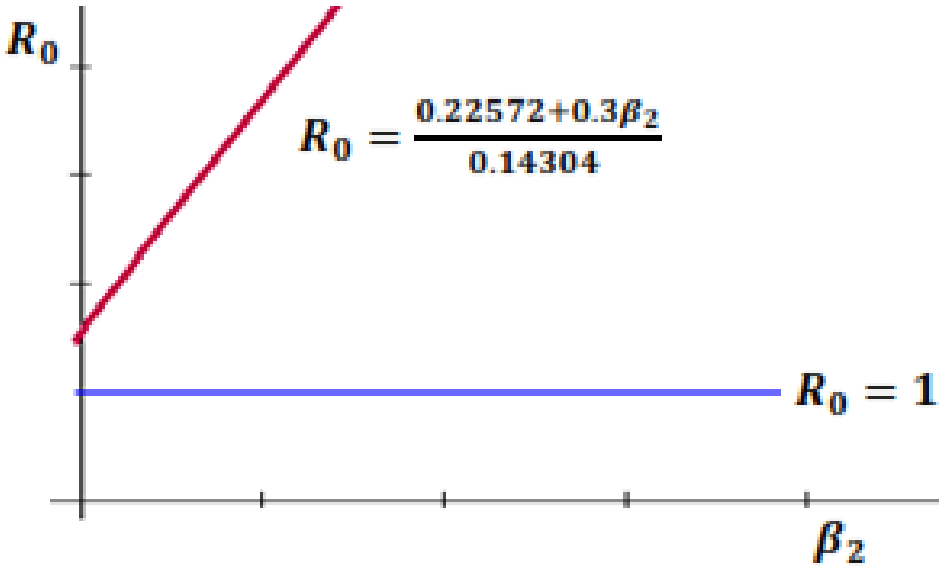


Figure 6.2: the relationship between the reproduction number and the rate of horizontal transmission of the disease from aware infective class.

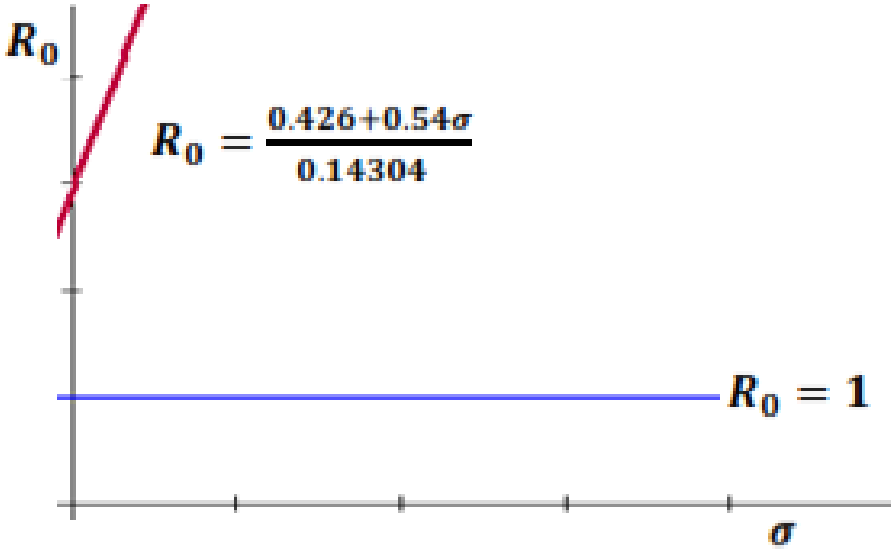


Figure 6.3: the relationship between the reproduction number and the rate of blood borne transmission of the disease.

From figure 6.3, we can observe that an increase in the rate of blood borne transmission, $\sigma$, makes an increase in the reproduction number, $R_0$. That is the disease always persists for any value of parameter $\sigma$.

From figure 6.4, we can observe that an increase in the rate of progress of unaware infective to AIDS, $\delta_1$, between the parametric values 0 and 1.49 makes a decrease in the reproduction

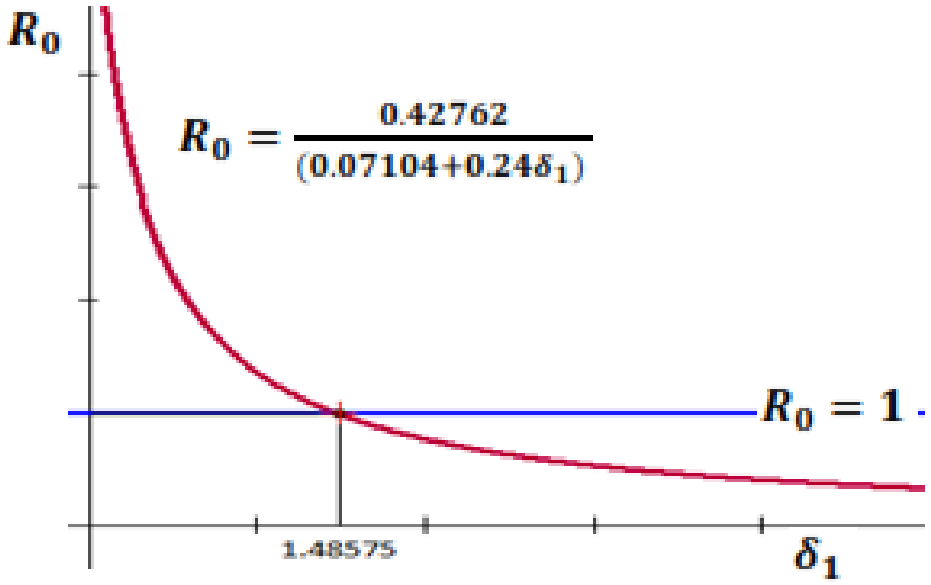


Figure 6.4: the relationship between the reproduction number and the rate of progress of unaware infective to AIDS.

number, $R_0$, but the reproduction number is greater than one that indicates the disease persists. If the parameter value of $\delta_1$ greater than 1.49, then the reproduction number decreases and becomes less than one where the disease dies out.

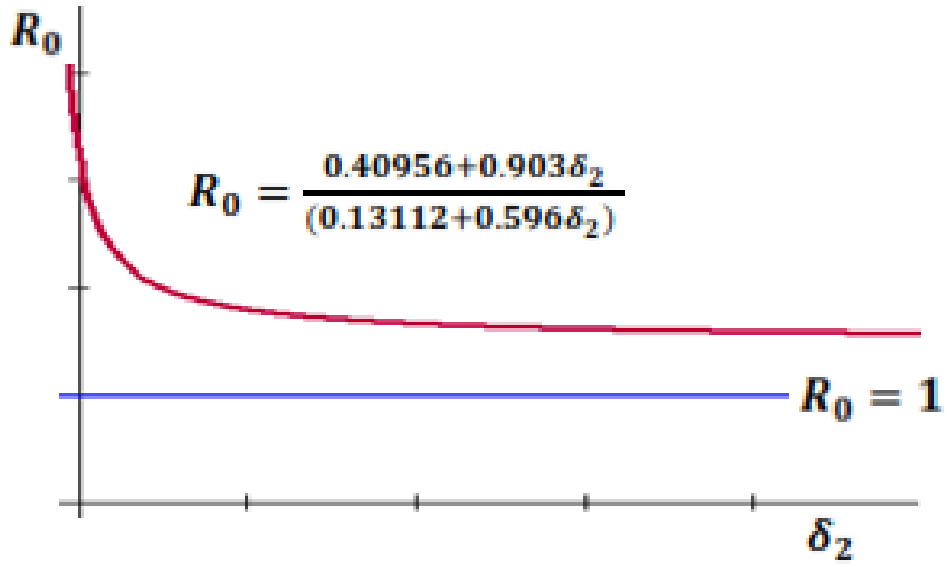


Figure 6.5: the relationship between the reproduction number and the rate of progress of aware infective to AIDS.

From figure 6.5, we can observe that an increase in the rate of progress of aware infective to AIDS, $\delta_2$, makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease still persists.

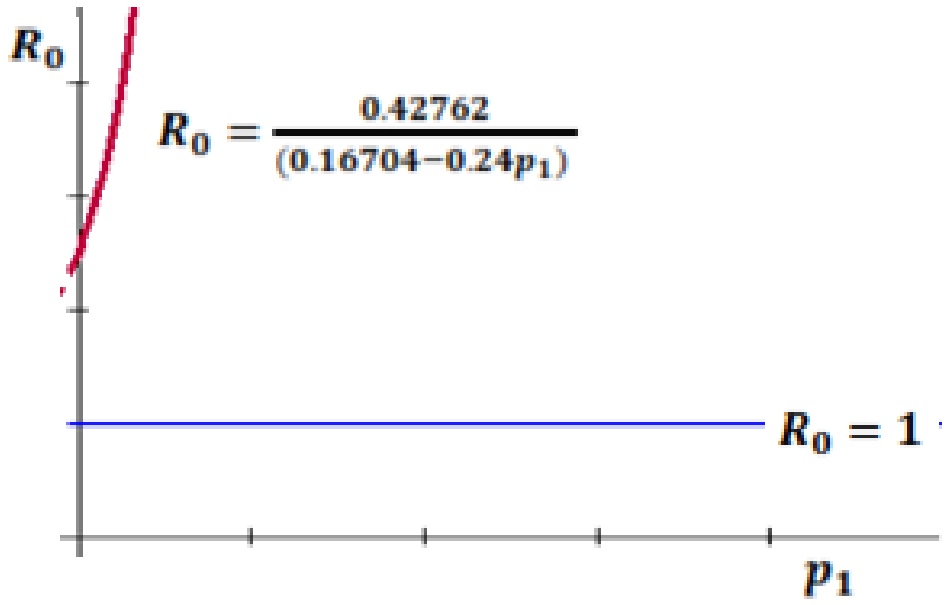


Figure 6.6: the relationship between the reproduction number and the rate of unaware infective immigrants.

From figure 6.6, we can observe that an increase in the rate of unaware infective immigrants, $p_1$, between 0 and 0.70, makes an increase in the reproduction number, $R_0 > 1$ and tell us the disease persists.

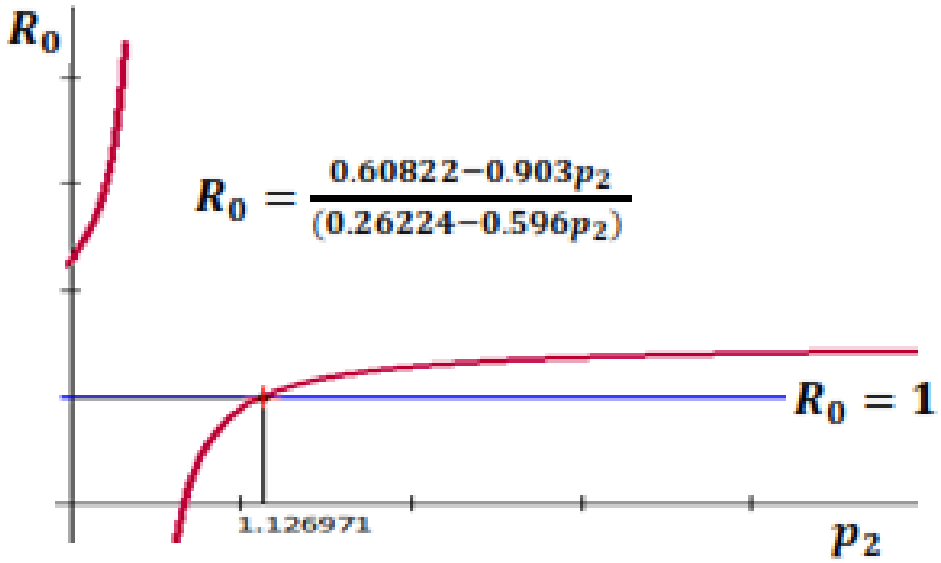


Figure 6.7: the relationship between the reproduction number and the rate of aware infective immigrants.

From figure 6.7, we can observe that an increase in the rate of aware infective immigrants,

$p_2$, between 0 and 0.44, makes an increase in the reproduction number, $R_0 > 1$ and tell us the disease persists. If the rate of aware infective immigrants between 0.44 and 1.13 makes an increase in the reproduction number with, $R_0 < 1$ and tell us the disease not persists. Whereas, the rate of aware infective immigrants greater than 1.13, makes an increase in the reproduction number, $R_0 > 1$, and tell us the disease persists.

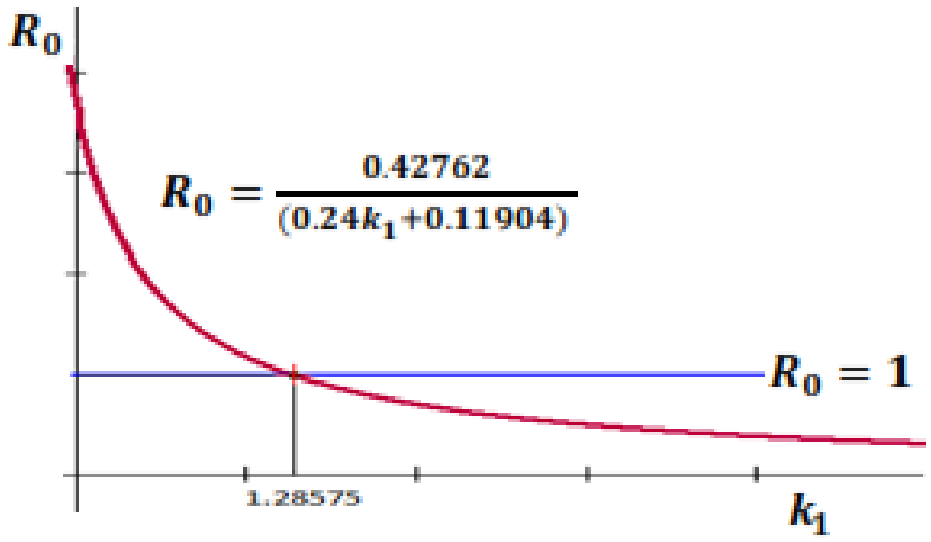


Figure 6.8: the relationship between the reproduction number and the rate of transmission of unaware infective to seropositive class.

From figure 6.8, we can observe that an increase in the rate of transmission of unaware infective to seropositive class, $k_1$, between 0 and 1.29, makes a decrease in the reproduction number, with $R_0 > 1$ and tell us the disease persists. If the rate of transmission of unaware infective to seropositive class greater than 1.29 makes a decrease in the reproduction number with, $R_0 < 1$ and tell us the disease dies out.

From figure 6.9, we can observe that an increase in the rate of transmission of aware infective to seropositive class, $k_2$, makes a decrease in the reproduction number, with $R_0 > 1$ and tell us the disease still persists.

From figure 6.10, we can observe that an increase in the rate of transmission of unaware infective to aware infective, $\theta$, then the reproduction number almost constant, with $R_0 > 1$ and tell us the disease still persists with constant reproduction number(i.e approximately 2.93 ).

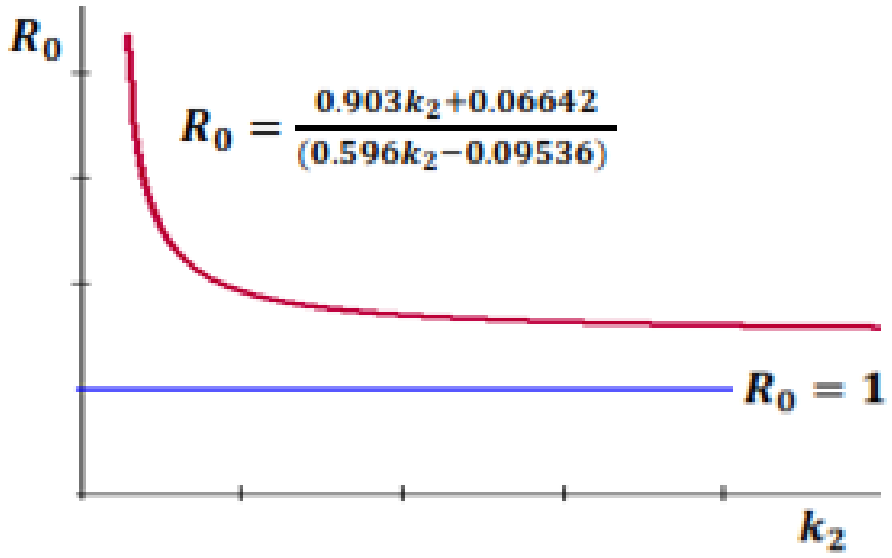


Figure 6.9: the relationship between the reproduction number and the rate of transmission of aware infective to seropositive class.

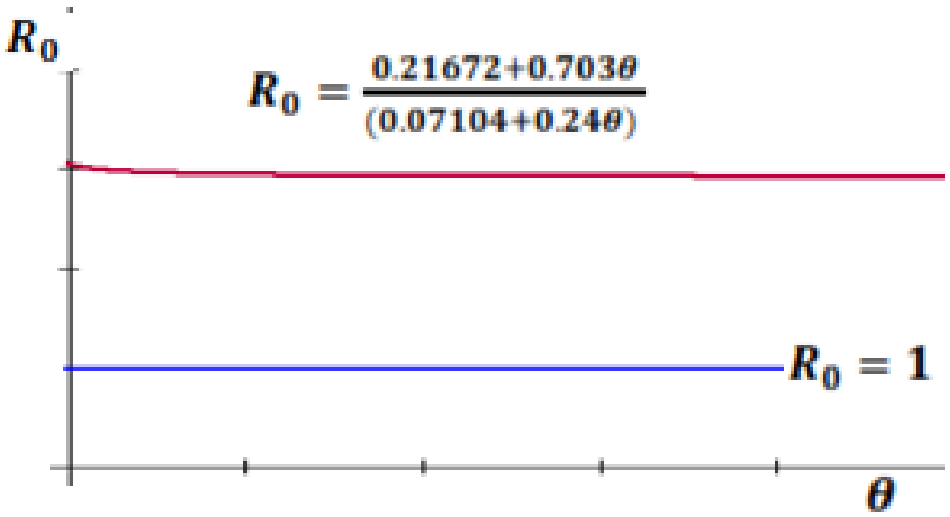


Figure 6.10: the relationship between the reproduction number and the rate of transmission of unaware infective to aware infective class.

From figure 6.11, we can observe that an increase in the rate of vertical transmission,$\phi$, between 0 and 0.78 then the reproduction number also increases, with $R_0 > 1$ and tell us the disease persists.

From figure 6.12, we can observe that an increase in the natural death rate, $\mu$, between 0 and 0.59 then the reproduction number decreases, with $R_0 > 1$ and tell us the disease still persists. If the natural death rate is greater than 0.59, then the reproduction number is decreases, with $R_0 < 1$ and this tell us the disease dies out.

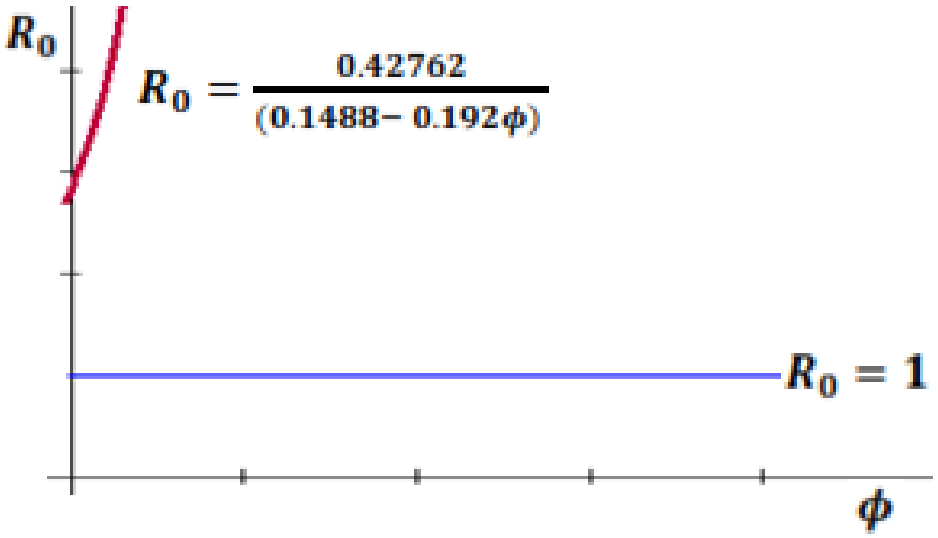


Figure 6.11: the relationship between the reproduction number and the rate of vertical transmission.

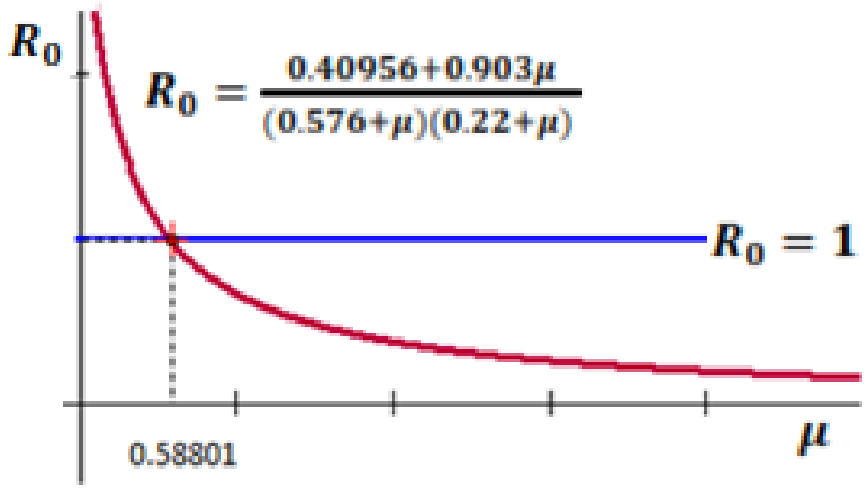


Figure 6.12: the relationship between the reproduction number and the natural death rate.

## 6.3 Results and Discussion

In this section, we would like to present the results and findings obtained from the analysis of the model.

From sensitive analysis we observed that the most sensitive parameter is the probability of the disease transmits to susceptible people by unaware infective humans, $\beta_1$ and the least sensitive parameter is the rate of transmission through blood borne,$\sigma$. The indices having positive signs increase the value of $R_0$ as one increase them and those having negative signs decrease the value of $R_0$, when they are increased.

Results from Numerical simulation show that as the probability of transmission of the disease from unaware infective and aware infective increases, the basic reproduction number increases. This will result in increasing on the transmission of HIV/AIDS.

We can observe that an increase in the rate of blood borne transmission, $\sigma$, makes an increase in the reproduction number, $R_0$. That is the disease always persists for any value of parameter $\sigma$.

An increase in the rate of progress of unaware infective to AIDS, $\delta_1$, between the parametric values 0 and 1.49 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists. If the parameter value of $\delta_1$ greater than 1.49, then the reproduction number decreases and becomes less than one where the disease dies out.

An increase in the rate of progress of aware infective to AIDS, $\delta_2$, makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease still persists.

We also observed that an increase in the rate of unaware infective immigrants, $p_1$, between 0 and 0.70, makes an increase in the reproduction number, $R_0 > 1$ and tell us the disease persists. An increase in the rate of aware infective immigrants, $p_2$, between 0 and 0.44, makes an increase in the reproduction number, $R_0 > 1$ and tell us the disease persists. If the rate of aware infective immigrants between 0.44 and 1.13 makes an increase in the reproduction number with, $R_0 < 1$ and tell us the disease dies out. Whereas, the rate of aware infective immigrants greater than 1.13, makes an increase in the reproduction number, $R_0 > 1$, and tell us the disease persists.

We can also observed that an increase in the rate of transmission of unaware infective to seropositive class, $k_1$, between 0 and 1.29, makes a decrease in the reproduction number, with $R_0 > 1$ and tell us the disease persists. If the rate of transmission of unaware infective to seropositive class greater than 1.29 makes a decrease in the reproduction number with, $R_0 < 1$ and tell us the disease dies out.

An increase in the rate of transmission of aware infective to seropositive class, $k_2$, makes a decrease in the reproduction number, with $R_0 > 1$ and tell us the disease still persists.

We also observed that an increase in the rate of transmission of unaware infective to aware infective, $\theta$, then the reproduction number almost constant, with $R_0 > 1$ and tell us the disease still persists with constant reproduction number(i.e approximately 2.93 ).

An increase in the rate of vertical transmission, $\phi$, between 0 and 0.78 then the reproduction number also increases, with $R_0 > 1$ and tell us the disease persists.

We can also observed that an increase in the natural death rate, $\mu$, between 0 and 0.59 then the

reproduction number decreases, with $R_0 > 1$ and tell us the disease still persists. If the natural death rate is greater than 0.59, then the reproduction number is decreases, with $R_0 < 1$ and this tell us the disease dies out.

## 6.4 Conclusions

We proposed an improvement of the model [104] that is to show the effect of unaware infective immigrants, aware infective immigrants, vertical and blood borne transmission and treatment on the dynamics of HIV/AIDS. A non-linear differential equation was formulated to represent the model. The stability analysis on the model shows that the disease free equilibrium point ($E_0$) is shown to be locally asymptotically stable and globally asymptotically stable when $R_0 < 1$and the positive endemic equilibrium point ($E^*$) is shown to be locally asymptotically stable and globally asymptotically stable when $R_0 > 1$. A sensitivity analysis of the basic reproduction number indicates that transmission probability, the rate of progress to AIDS and the rate of aware infective immigrants are the most sensitive parameters that can be used to control the spread of the disease. Results from Numerical simulation show that as the probability of transmission of the disease to susceptible individuals by unaware and aware infective individuals increases, the basic reproduction number also increases. This will result in increasing on the transmission of HIV/AIDS.

# Chapter 7

# Parameter estimation, Sensitivity analysis and Numerical simulations on the Dynamics of HIV/AIDS with Age Structure and Inflow Immigrants in Ethiopia

This chapter discussed about Parameter estimation, Sensitivity analysis and Numerical simulations of the model developed and analyzed under chapter five.

## 7.1 Parameter estimation

Estimation theory is a branch of statistics that deals with estimating the values of parameters based on measured empirical data that has a random component. The parameters describe an underlying physical setting in such a way that their value affects the distribution of the measured data. The term parameter estimation refers to the process of using sample data to estimate the parameters of the selected distribution. In this paper, we analyzed a non-linear mathematical $S_1S_2I_1I_2PS_PA$ model of HIV virus with horizontal, vertical, and blood born transmissions using the secondary data obtained from Ministry of Health of Federal Republic of Ethiopia, Central Statistical Agency (CSA) and related materials. To study the spread and

control of HIV/AIDS in Ethiopia, we divide the total population in to seven compartments, such as $S_1$: sexually mature susceptible age greater than or equal to 15 years, $S_2$: sexually immature susceptible age less than 15, $I_1$: unaware infective, $I_2$: aware infective, $P$: Pre-AIDS, $S_p$: seropositive/treatment and $A$: AIDS classes. We obtained secondary data from the reports of world Health Organization (WHO), Federal Democratic Republic of Ethiopia Ministry of Health and related materials.

Table 7.1: Ethiopia Demographics Profile 2019 in gender[47]

| **Description** | **Notation** | **Values** |
|---|---|---|
| Total number of Female in Ethiopia | F | 54,550,770 |
| Total number of Male in Ethiopia | M | 53,835,621 |
| Total number of population in Ethiopia | T | 108,386,391 |

Table 7.2: Ethiopia Demographics Profile 2019 in ages[47]

| **Description** | **Values** |
|---|---|
| Total number of people under 15 years old | 41,831,101 |
| Total number of people 15 years old and above | 61,555,290 |
| Total number of population in Ethiopia | 108,386,391 |

In this paper the total population is $N = S_1 + S_2 + I_1 + I_2 + P + S_p + A$. The parameter values of the present model are obtained as:

$Q_1$ =average number of susceptible immigrants arriving to a country= $230,000/year$

$Q_2$ =birth rate× sexually immature population= $\frac{(0.79\times 41,831,101)}{100} = 330465/year$

$\beta_1 = \frac{\text{Effective contact of unaware infective}}{\text{Total contact of unaware infective}} = 0.83$

$\beta_2 = \frac{\text{Effective contact of aware infective}}{\text{Total contact of aware infective}} = 0.7$

$\beta_3 = \frac{\text{Effective contact of Pre-AIDS}}{\text{Total contact of Pre-AIDS individuals}} = 0.9$

$\sigma = \frac{\text{Average number of infective by blood born transmission per year}}{\text{Total number of infective}} = 0.03$

$\mu$ = Natural death rate = 0.0065

$\theta = \frac{\text{number of unaware infected who know their status per year}}{\text{total number of unaware infective}} = 0.79$

$\delta_1 = \frac{1}{\text{Average life time of unaware infective individual progress to AIDS}} = 0.06$

$\delta_2 = \frac{1}{\text{Average life time of aware infective individual progress to AIDS}} = 0.06$

$$\delta_3 = \frac{1}{\text{Average life time of Pre-AIDS individual progress to AIDS}} = 0.4621$$

$$k_1 = \frac{\text{Average number of treated unaware HIV infected per year}}{\text{Total number unaware HIV infected population}} = 0.02$$

$$k_2 = \frac{\text{Average number of treated aware HIV infected per year}}{\text{Total number of aware HIV infected population}} = 0.65$$

$$k_3 = \frac{\text{Average number of treated Pre-AIDS individuals per year}}{\text{Total number of Pre-AIDS population}} = 0.65$$

$$p_1 = \frac{\text{Average number of unaware HIV infective immigrants per year}}{\text{Tota number of infective}} = 0.016$$

$$p_2 = \frac{\text{Average number of aware HIV infective immigrants per year}}{\text{Tota number of infective}} = 0.013$$

$$u_1 = \frac{1}{\text{Average life time of unaware infective individual progress to Pre-AIDS}} = 0.36$$

$$u_2 = \frac{1}{\text{Average life time of aware infective individual progress to Pre-AIDS}} = 0.57$$

$$\phi = \frac{\text{Average number of infected new born per year}}{\text{Total number of new born}} = 0.45$$

$$\epsilon = \frac{\text{Average number of new born death per year}}{\text{Total number of birth}} = 0.0281$$

$$\alpha = \frac{\text{Average number of AIDS population died due to the disease per year}}{\text{Total number of infective}} = 0.0159$$

Table 7.3: Summary of Parameter values

| **Parameter** | **Parameter Description** | **Value** | **Data Source** |
|---|---|---|---|
| $Q_1$ | Recruitment in to sexual mature population | 230000 | [106] |
| $Q_2$ | Recruitment in to sexual immature population | 330465 | [47] |
| $\beta_1$ | The horizontal transmission rate of unaware infective to susceptible individuals | 0.83 | [29] |
| $\beta_2$ | The horizontal transmission rate of aware infective to susceptible individuals | 0.7 | [29] |
| $\beta_3$ | The horizontal transmission rate of Pre-AIDS to susceptible individuals | 0.9 | [29] |
| $\sigma$ | Rate of transmission through blood borne | 0.03 | [74] |
| $\delta_1$ | Rate at which unaware infective develop full blown AIDS | 0.06 | [46] |
| $\delta_2$ | Rate at which aware infective develop full blown AIDS | 0.06 | [46] |
| $\delta_3$ | Progression rate of Pre-AIDS individuals to full blown AIDS | 0.4621 | [110] |
| $\mu$ | Natural mortality | 0.0065 | [48] |
| $\theta$ | Rate of status awareness due to screening method | 0.79 | [49] |

| | | | |
|---|---|---|---|
| $k_1$ | Rate of treatment of unaware infective | 0.02 | [49] |
| $k_2$ | Rate of treatment of aware infective | 0.65 | [49] |
| $k_3$ | Rate of treatment of Pre-AIDS | 0.65 | [49] |
| $p_1$ | Rate of unaware infective immigrants | 0.016 | [106] |
| $p_2$ | Rate of aware infective immigrants | 0.013 | [106] |
| $u_1$ | Rate of progress to Pre-AIDS from unaware infective | 0.36 | [110] |
| $u_2$ | Rate of progress to Pre-AIDS from aware infective | 0.57 | [110] |
| $\phi$ | Rate of vertical transmission | 0.45 | [30] |
| $\epsilon$ | Probability of death at birth | 0.0281 | [45] |
| $\alpha$ | AIDS induced death rate | 0.0159 | [47] |

### 7.1.1 Sensitivity analysis

The parameter values and assumptions of any model are subject to change and error. Sensitivity analysis is the investigation of these potential changes & errors and their impacts on conclusions to be drawn from the model. Here we use it to discover parameters that have a high impact on reproduction number $R_0$. We calculate the sensitivity indices of the reproductive number $R_0$, to the parameters in the model. These indices tell us how crucial each parameter is to disease transmission and prevalence. The normalized forward sensitivity index of a variable to a parameter is the ratio of the relative change in the variable to the relative change in the parameter. When the variable is a differentiable function of the parameter, the sensitivity index may be alternatively defined using partial derivatives [78].

**Definition**: The normalized forward sensitivity index of a variable, $u$ that depends differentiable on a parameter $p$, is defined as:

$$SI(p) = \frac{\partial u}{\partial p} \times \frac{p}{u}$$

If the magnitude of sensitivity index is high for the parameter $p$ out of other parameters then we say that $p$ is more sensitive parameter.

Here we consider parameters $Q_1, Q_2, \beta_1, \beta_2, \beta_3, \sigma, \delta_1, \delta_2, \delta_3, \theta, k_1, k_2, k_3, \phi, p_1, p_2, u_1, u_2, \epsilon$ and $\mu$ to see the sensitivity parameter with regard to basic reproduction number $R_0$ as follows:

$$R_0 = (\beta_1 \frac{Q_1}{Q_1+Q_2} + \sigma)\frac{1}{\nabla_1} + (\beta_2 \frac{Q_1}{Q_1+Q_2} + \sigma)\frac{\theta}{\nabla_1\nabla_2} + (\beta_3 \frac{Q_1}{Q_1+Q_2} + \sigma)\frac{\theta u_2 + u_1\nabla_2}{\nabla_1\nabla_2\nabla_3} = 1.05$$

$$SI\left(Q_1\right)=\frac{\partial R_0}{\partial Q_1}x\frac{Q_1}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial Q_1}\right]\left[\frac{Q_1}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial Q_1}\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1}+\frac{\partial}{\partial Q_1}\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2}+\frac{\partial}{\partial Q_1}\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{Q_1}{R_0}\right]$$

$$=\left[\frac{\beta_1Q_2}{\nabla_1(Q_1+Q_2)^2}+\frac{\beta_2Q_2\theta}{\nabla_1\nabla_2(Q_1+Q_2)^2}+\frac{\beta_3Q_2(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3(Q_1+Q_2)^2}\right]\left[\frac{Q_1}{R_0}\right]$$

$$=\frac{1}{(Q_1+Q_2)^2}\left[\frac{\beta_1Q_2}{\nabla_1}+\frac{\beta_2Q_2\theta}{\nabla_1\nabla_2}+\frac{\beta_3Q_2(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right]\left[\frac{Q_1}{R_0}\right]=0.00$$

$$SI\left(Q_2\right)=\frac{\partial R_0}{\partial Q_2}x\frac{Q_2}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial Q_2}\right]\left[\frac{Q_2}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial Q_2}\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1}+\frac{\partial}{\partial Q_2}\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2}+\frac{\partial}{\partial Q_2}\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{Q_2}{R_0}\right]$$

$$=\left[\frac{-\beta_1Q_1}{\nabla_1(Q_1+Q_2)^2}+\frac{-\beta_2Q_1\theta}{\nabla_1\nabla_2(Q_1+Q_2)^2}+\frac{-\beta_3Q_1(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3(Q_1+Q_2)^2}\right]\left[\frac{Q_2}{R_0}\right]$$

$$=\frac{-1}{(Q_1+Q_2)^2}\left[\frac{\beta_1Q_1}{\nabla_1}+\frac{\beta_2Q_1\theta}{\nabla_1\nabla_2}+\frac{\beta_3Q_1(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right]\left[\frac{Q_2}{R_0}\right]=0.00$$

$$SI\left(\beta_1\right)=\frac{\partial R_0}{\partial\beta_1}x\frac{\beta_1}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial\beta_1}\right]\left[\frac{\beta_1}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial\beta_1}\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1}+\frac{\partial}{\partial\beta_1}\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2}+\frac{\partial}{\partial\beta_1}\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{\beta_1}{R_0}\right]$$

$$=\left[\frac{Q_1}{\nabla_1(Q_1+Q_2)}\right]\left[\frac{\beta_1}{R_0}\right]=0.41$$

$$SI\left(\beta_2\right)=\frac{\partial R_0}{\partial\beta_2}x\frac{\beta_2}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial\beta_2}\right]\left[\frac{\beta_2}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial\beta_2}\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1}+\frac{\partial}{\partial\beta_2}\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2}+\frac{\partial}{\partial\beta_2}\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{\beta_2}{R_0}\right]$$

$$=\left[\frac{\theta Q_1}{\nabla_1\nabla_2(Q_1+Q_2)}\right]\left[\frac{\beta_2}{R_0}\right]=0.22$$

$$SI\left(\beta_3\right)=\frac{\partial R_0}{\partial\beta_3}x\frac{\beta_3}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial\beta_3}\right]\left[\frac{\beta_3}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial\beta_3}\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1}+\frac{\partial}{\partial\beta_3}\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2}+\frac{\partial}{\partial\beta_3}\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{\beta_3}{R_0}\right]$$

$$=\left[\frac{(\theta u_2+u_1\nabla_2)Q_1}{\nabla_1\nabla_2\nabla_3(Q_1+Q_2)}\right]\left[\frac{\beta_3}{R_0}\right]=0.29$$

$$SI\left(\sigma\right)=\frac{\partial R_0}{\partial\sigma}x\frac{\sigma}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial\sigma}\right]\left[\frac{\sigma}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial\sigma}\left(\beta_1\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1}+\frac{\partial}{\partial\sigma}\left(\beta_2\frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2}+\frac{\partial}{\partial\sigma}\left(\beta_3\frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{\sigma}{R_0}\right]$$

$$=\left[\frac{1}{\nabla_1}+\frac{\theta}{\nabla_1\nabla_2}+\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right]\left[\frac{\sigma}{R_0}\right]$$

$$=\left[\frac{\nabla_2\nabla_3+\theta\nabla_3+\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right]\left[\frac{\sigma}{R_0}\right]=0.08$$

$$SI\left(\delta_1\right)=\frac{\partial R_0}{\partial \delta_1} x \frac{\delta_1}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \nabla_2 \nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta \nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\theta u_2+u_1 \nabla_2\right)}{\nabla_1 \nabla_2 \nabla_3}\right)}{\partial \delta_1}\right]\left[\frac{\delta_1}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial \delta_1}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{1}{\nabla_1}+\frac{\partial}{\partial \delta_1}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{\theta}{\nabla_1 \nabla_2}+\frac{\partial}{\partial \delta_1}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1 \nabla_2}{\nabla_1 \nabla_2 \nabla_3}\right)\right]\left[\frac{\delta_1}{R_0}\right]$$

$$=\left[-\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{1}{{\nabla_1}^2}-\frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta}{\nabla_2 {\nabla_1}^2}-\frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\theta u_2+u_1 \nabla_2\right)}{\nabla_2 \nabla_3 {\nabla_1}^2}\right]\left[\frac{\delta_1}{R_0}\right]$$

$$=\left[-\frac{R_0}{\nabla_1}\right]\left[\frac{\delta_1}{R_0}\right]=-\frac{\delta_1}{\nabla_1}=-0.08$$

$$SI\left(\delta_2\right)=\frac{\partial R_0}{\partial \delta_2} x \frac{\delta_2}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \nabla_2 \nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta \nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\theta u_2+u_1 \nabla_2\right)}{\nabla_1 \nabla_2 \nabla_3}\right)}{\partial \delta_2}\right]\left[\frac{\delta_2}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial \delta_2}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{1}{\nabla_1}+\frac{\partial}{\partial \delta_2}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{\theta}{\nabla_1 \nabla_2}+\frac{\partial}{\partial \delta_2}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1 \nabla_2}{\nabla_1 \nabla_2 \nabla_3}\right)\right]\left[\frac{\delta_2}{R_0}\right]$$

$$=\left[-\frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta}{\nabla_1 {\nabla_2}^2}-\frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta u_2}{\nabla_1 \nabla_3 {\nabla_2}^2}\right]\left[\frac{\delta_2}{R_0}\right]$$

$$=-\left[\frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta \nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta u_2}{\nabla_1 \nabla_3 {\nabla_2}^2}\right]\left[\frac{\delta_2}{R_0}\right]=-0.02$$

$$SI\left(\delta_3\right)=\frac{\partial R_0}{\partial \delta_3} x \frac{\delta_3}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \nabla_2 \nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta \nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\theta u_2+u_1 \nabla_2\right)}{\nabla_1 \nabla_2 \nabla_3}\right)}{\partial \delta_3}\right]\left[\frac{\delta_3}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial \delta_3}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{1}{\nabla_1}+\frac{\partial}{\partial \delta_3}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{\theta}{\nabla_1 \nabla_2}+\frac{\partial}{\partial \delta_3}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1 \nabla_2}{\nabla_1 \nabla_2 \nabla_3}\right)\right]\left[\frac{\delta_3}{R_0}\right]$$

$$=\left[-\frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\theta u_2+u_1 \nabla_2\right)}{\nabla_1 \nabla_2 {\nabla_3}^2}\right]\left[\frac{\delta_3}{R_0}\right]$$

$$=-\left[\frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\theta u_2+u_1 \nabla_2\right)}{\nabla_1 \nabla_2 {\nabla_3}^2}\right]\left[\frac{\delta_3}{R_0}\right]=-0.13$$

$$SI\left(\theta\right)=\frac{\partial R_0}{\partial \theta} x \frac{\theta}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \nabla_2 \nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta \nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\theta u_2+u_1 \nabla_2\right)}{\nabla_1 \nabla_2 \nabla_3}\right)}{\partial \theta}\right]\left[\frac{\theta}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial \theta}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{1}{\nabla_1}+\frac{\partial}{\partial \theta}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{\theta}{\nabla_1 \nabla_2}+\frac{\partial}{\partial \theta}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1 \nabla_2}{\nabla_1 \nabla_2 \nabla_3}\right)\right]\left[\frac{\theta}{R_0}\right]$$

$$=\left[\frac{\left[\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right) u_2\right] \nabla_1 \nabla_2 \nabla_3-R_0 \nabla_1 {\nabla_2}^2 {\nabla_3}^2}{\left(\nabla_1 \nabla_2 \nabla_3\right)^2}\right]\left[\frac{\theta}{R_0}\right]$$

$$=\left[\frac{\left[\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right) u_2\right]}{\nabla_1 \nabla_2 \nabla_3}-\frac{R_0}{\nabla_1}\right]\left[\frac{\theta}{R_0}\right]=-0.62$$

$$SI\left(k_1\right)=\frac{\partial R_0}{\partial k_1} x \frac{k_1}{R_0}=\left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \nabla_2 \nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta \nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\theta u_2+u_1 \nabla_2\right)}{\nabla_1 \nabla_2 \nabla_3}\right)}{\partial k_1}\right]\left[\frac{k_1}{R_0}\right]$$

$$=\left[\frac{\partial}{\partial k_1}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{1}{\nabla_1}+\frac{\partial}{\partial k_1}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{\theta}{\nabla_1 \nabla_2}+\frac{\partial}{\partial k_1}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1 \nabla_2}{\nabla_1 \nabla_2 \nabla_3}\right)\right]\left[\frac{k_1}{R_0}\right]$$

$$=\left[-\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \frac{1}{{\nabla_1}^2}-\frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right) \theta}{\nabla_2 {\nabla_1}^2}-\frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\theta u_2+u_1 \nabla_2\right)}{\nabla_2 \nabla_3 {\nabla_1}^2}\right]\left[\frac{k_1}{R_0}\right]$$

$$= \left[-\frac{R_0}{\nabla_1}\right]\left[\frac{k_1}{R_0}\right] = -\frac{k_1}{\nabla_1} = -0.03$$

$$SI\left(k_2\right) = \frac{\partial R_0}{\partial k_2} x \frac{k_2}{R_0} = \left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial k_2}\right]\left[\frac{k_2}{R_0}\right]$$

$$= \left[\frac{\partial}{\partial k_2}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1} + \frac{\partial}{\partial k_2}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2} + \frac{\partial}{\partial k_2}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{k_2}{R_0}\right]$$

$$= \left[-\frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta}{\nabla_1{\nabla_2}^2} - \frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta u_2}{\nabla_1\nabla_3{\nabla_2}^2}\right]\left[\frac{k_2}{R_0}\right]$$

$$= -\left[\frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta u_2}{\nabla_1\nabla_3{\nabla_2}^2}\right]\left[\frac{k_2}{R_0}\right] = -0.20$$

$$SI\left(k_3\right) = \frac{\partial R_0}{\partial k_3} x \frac{k_3}{R_0} = \left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial k_3}\right]\left[\frac{k_3}{R_0}\right]$$

$$= \left[\frac{\partial}{\partial k_3}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1} + \frac{\partial}{\partial k_3}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2} + \frac{\partial}{\partial k_3}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{k_3}{R_0}\right]$$

$$= \left[-\frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2{\nabla_3}^2}\right]\left[\frac{k_3}{R_0}\right]$$

$$= -\left[\frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2{\nabla_3}^2}\right]\left[\frac{k_3}{R_0}\right] = -0.18$$

$$SI\left(\phi\right) = \frac{\partial R_0}{\partial \phi} x \frac{\phi}{R_0} = \left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial \phi}\right]\left[\frac{\phi}{R_0}\right]$$

$$= \left[\frac{\partial}{\partial \phi}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1} + \frac{\partial}{\partial \phi}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2} + \frac{\partial}{\partial \phi}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{\phi}{R_0}\right]$$

$$= \left[\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{(1-\epsilon)}{{\nabla_1}^2} + \frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta(1-\epsilon)}{\nabla_2{\nabla_1}^2} + \frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)(1-\epsilon)}{\nabla_2\nabla_3{\nabla_1}^2}\right]\left[\frac{\phi}{R_0}\right]$$

$$= \left[\frac{(1-\epsilon)R_0}{\nabla_1}\right]\left[\frac{\phi}{R_0}\right] = \frac{(1-\epsilon)\phi}{\nabla_1} = 0.56$$

$$SI\left(p_1\right) = \frac{\partial R_0}{\partial p_1} x \frac{p_1}{R_0} = \left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial p_1}\right]\left[\frac{p_1}{R_0}\right]$$

$$= \left[\frac{\partial}{\partial p_1}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1} + \frac{\partial}{\partial p_1}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2} + \frac{\partial}{\partial p_1}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{p_1}{R_0}\right]$$

$$= \left[\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{{\nabla_1}^2} + \frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta}{\nabla_2{\nabla_1}^2} + \frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_2\nabla_3{\nabla_1}^2}\right]\left[\frac{p_1}{R_0}\right]$$

$$= \left[\frac{R_0}{\nabla_1}\right]\left[\frac{p_1}{R_0}\right] = \frac{p_1}{\nabla_1} = 0.02$$

$$SI\left(p_2\right) = \frac{\partial R_0}{\partial p_2} x \frac{p_2}{R_0} = \left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial p_2}\right]\left[\frac{p_2}{R_0}\right]$$

$$= \left[\frac{\partial}{\partial p_2}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1} + \frac{\partial}{\partial p_2}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2} + \frac{\partial}{\partial p_2}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{p_2}{R_0}\right]$$

$$= \left[\frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta}{\nabla_1{\nabla_2}^2} + \frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta u_2}{\nabla_1\nabla_3{\nabla_2}^2}\right]\left[\frac{p_2}{R_0}\right]$$

$$= \left[\frac{\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta u_2}{\nabla_1\nabla_3{\nabla_2}^2}\right]\left[\frac{p_2}{R_0}\right] = 0.00$$

$$SI\left(u_1\right) = \frac{\partial R_0}{\partial u_1} x \frac{u_1}{R_0} = \left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial u_1}\right]\left[\frac{u_1}{R_0}\right]$$

$$= \left[\frac{\partial}{\partial u_1}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1} + \frac{\partial}{\partial u_1}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2} + \frac{\partial}{\partial u_1}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{u_1}{R_0}\right]$$

$$= \left[\frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_1\nabla_2\nabla_3 - R_0\nabla_1{\nabla_2}^2{\nabla_3}^2}{\left(\nabla_1\nabla_2\nabla_3\right)^2}\right]\left[\frac{u_1}{R_0}\right]$$

$$= \left[\frac{\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)}{\nabla_1\nabla_3} - \frac{R_0}{\nabla_1}\right]\left[\frac{u_1}{R_0}\right] = -0.30$$

$$SI\left(u_2\right) = \frac{\partial R_0}{\partial u_2} x \frac{u_2}{R_0} = \left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial u_2}\right]\left[\frac{u_2}{R_0}\right]$$

$$= \left[\frac{\partial}{\partial u_2}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1} + \frac{\partial}{\partial u_2}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2} + \frac{\partial}{\partial u_2}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{u_2}{R_0}\right]$$

$$= \left[\frac{\left[\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta+u_2)\right]\nabla_1\nabla_2\nabla_3 - R_0\nabla_2\nabla_1{\nabla_2}^2{\nabla_3}^2}{\left(\nabla_1\nabla_2\nabla_3\right)^2}\right]\left[\frac{u_2}{R_0}\right]$$

$$= \left[\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta+u_2)}{\nabla_1\nabla_2\nabla_3} - \frac{R_0}{\nabla_2}\right]\left[\frac{u_2}{R_0}\right] = 0.07$$

$$SI\left(\epsilon\right) = \frac{\partial R_0}{\partial \epsilon} x \frac{\epsilon}{R_0} = \left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial \epsilon}\right]\left[\frac{\epsilon}{R_0}\right]$$

$$= \left[\frac{\partial}{\partial \epsilon}\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1} + \frac{\partial}{\partial \epsilon}\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2} + \frac{\partial}{\partial \epsilon}\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{\epsilon}{R_0}\right]$$

$$= \left[-\phi\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{{\nabla_1}^2} - \phi\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{{\nabla_1}^2\nabla_2} - \phi\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{{\nabla_1}^2\nabla_2\nabla_3}\right)\right]\left[\frac{\epsilon}{R_0}\right]$$

$$= \frac{-\phi}{\nabla_1}\left[\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{1}{\nabla_1} + \left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\frac{\theta}{\nabla_1\nabla_2} + \left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\left(\frac{\theta u_2+u_1\nabla_2}{\nabla_1\nabla_2\nabla_3}\right)\right]\left[\frac{\epsilon}{R_0}\right]$$

$$= \frac{-\phi}{\nabla_1} R_0 \times \frac{\epsilon}{R_0} = \frac{-\phi\epsilon}{\nabla_1} = -0.02$$

$$SI\left(\mu\right) = \frac{\partial R_0}{\partial \mu} x \frac{\mu}{R_0} = \left[\frac{\partial\left(\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\nabla_2\nabla_3+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta\nabla_3+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)(\theta u_2+u_1\nabla_2)}{\nabla_1\nabla_2\nabla_3}\right)}{\partial \mu}\right]\left[\frac{\mu}{R_0}\right]$$

$$= \left[\frac{\left[\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)[\nabla_3+\nabla_2]+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)u_1\right]\nabla_1\nabla_2\nabla_3 - R_0\nabla_1\nabla_2\nabla_3\left[\frac{\partial}{\partial \mu}(\nabla_1\nabla_2\nabla_3)\right]}{\left(\nabla_1\nabla_2\nabla_3\right)^2}\right]\left[\frac{\mu}{R_0}\right]$$

$$= \left[\frac{\left(\beta_1 \frac{Q_1}{Q_1+Q_2}+\sigma\right)[\nabla_3+\nabla_2]+\left(\beta_2 \frac{Q_1}{Q_1+Q_2}+\sigma\right)\theta+\left(\beta_3 \frac{Q_1}{Q_1+Q_2}+\sigma\right)u_1}{\nabla_1\nabla_2\nabla_3} - \frac{R_0}{\nabla_1\nabla_2\nabla_3}\left[\nabla_2\nabla_3+\nabla_1\nabla_2+\nabla_1\nabla_3\right]\right]\left[\frac{\mu}{R_0}\right]$$

$$= -0.01$$

We summarized values of sensitivity index for the important parameters by the table below.

Table 7.4: Sensitivity indices

| **Parameter** | **Sensitivity Index** |
|---|---|
| $\theta$ | -0.62 |
| $\phi$ | 0.56 |
| $\beta_1$ | 0.41 |
| $u_1$ | -0.30 |
| $\beta_3$ | 0.29 |
| $\beta_2$ | 0.22 |
| $k_2$ | -0.20 |
| $k_3$ | -0.18 |
| $\delta_3$ | -0.13 |
| $\sigma$ | 0.08 |
| $\delta_1$ | -0.08 |
| $u_2$ | 0.07 |
| $k_1$ | -0.03 |
| $p_1$ | 0.02 |
| $\delta_2$ | -0.02 |
| $\epsilon$ | -0.02 |
| $\mu$ | -0.01 |
| $p_2$ | 0.00 |
| $Q_1$ | 0.00 |
| $Q_2$ | 0.00 |

Table 7.4 contains positive and negative sensitivity indices. The parameters are ordered from most sensitive to least. The most sensitive parameter is the transmission rate of unaware infective humans to aware infective, $\theta$ and the least sensitive parameter is the recruitment into sexually immature class, $Q_2$. The indices having positive signs increase the value of $R_0$ as one increase them and those having negative signs decrease the value of $R_0$, when they are increased.

One of the most sensitive parameter is $\theta$. Increasing (or decreasing) $\theta$ by 10%, decreases (or increases) $R_0$ by 6.16%. If the sensitivity index of $\beta_1 = 0.414$ , increasing $\beta_1$ by 10%, increasing $R_0$ by 4.14%. And also if we see the sensitivity index of$k_2 = -0.201$, increasing $k_2$ by 10%, decreases $R_0$ by 2.01%. We can similarly interpret the remaining parameter values.

The effectiveness of varying each parameter can thus be determined. As regards HIV/AIDS control, interventions must target the most sensitive parameters. For instance, interventions that target HIV status awareness and person to person transmission of the disease are the most effective in controlling the disease.

### 7.1.2 Numerical Simulations

The numerical analysis is obtained from the graphs of basic reproduction number with respect to the parameters obtained and listed in table 7.4

**Recruitment in to sexually mature and immature population.**

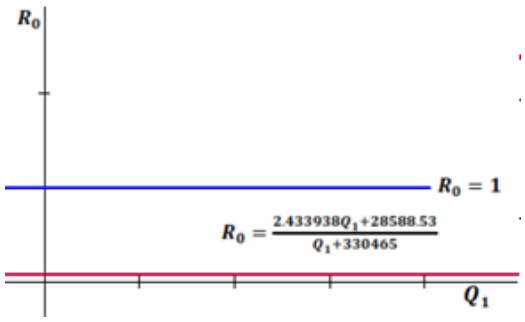


Figure 7.1: Reproduction number versus recruitment in to sexually mature population.

From figure 7.1, we can observe that an increase in recruitment in to sexually mature population, $Q_1$, makes the reproduction number, $R_0$ always below one. That is the disease not persists for any value of parameter $Q_1$.

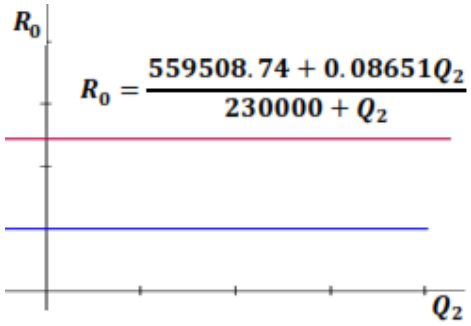


Figure 7.2: Reproduction number versus Recruitment in to sexually immature population.

From figure 7.2, we can observe that an increase in recruitment in to sexually immature population, $Q_2$, makes the reproduction number, $R_0$ always above one. That is the disease always

persists for any value of parameter $Q_2$ at constant reproduction number$R_0$.

**Rate of transmission of the disease from unaware and aware infective classes**

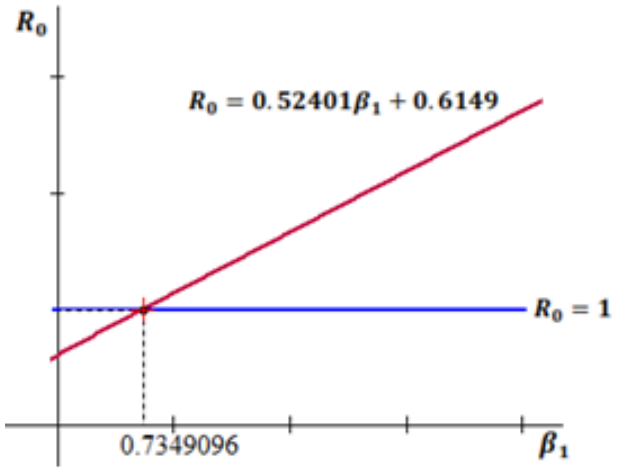


Figure 7.3: Reproduction number versus the rate of horizontal transmission of unaware infective

Figure 7.3: it is graphical representation of the basic reproduction number $R_0$ versus rate of horizontal transmission of the disease from unaware infective class $\beta_1$ and keeping other parameters constant. This figure shows that an increase in the rate of horizontal transmission of unaware infective, $\beta_1$, makes an increase in the reproduction number, $R_0$. If $\beta_1 > 0.73$ the reproduction number $R_0 > 1$ that indicates the disease persists. When $\beta_1 < 0.73$ the reproduction number $R_0 < 1$ this indicates that the disease not persists.

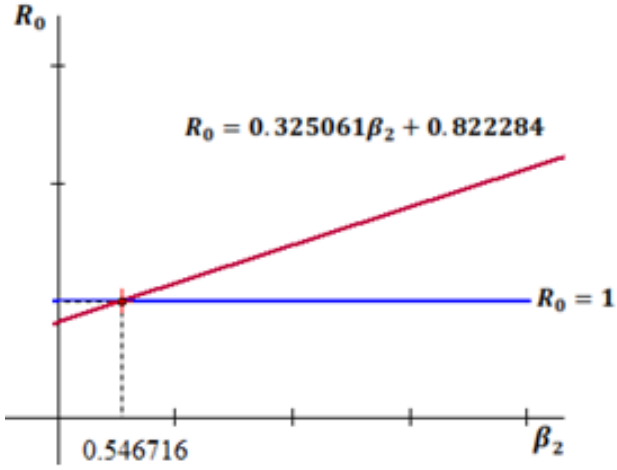


Figure 7.4: Reproduction number versus the rate of horizontal transmission of aware infective

Figure 7.4: it is graphical representation of the basic reproduction number $R_0$ versus rate of horizontal transmission of the disease from aware infective class $\beta_2$ and keeping other parameters constant. This figure shows that an increase in the rate of horizontal transmission of aware

infective, $\beta_2$, makes an increase in the reproduction number, $R_0$. If $\beta_2 > 0.55$ the reproduction number $R_0 > 1$ that indicates the disease persists. When $\beta_2 < 0.55$ the reproduction number $R_0 < 1$ this indicates that the disease not persists.

**Rate of transmission of the disease from Pre-AIDS class and blood born transmission**

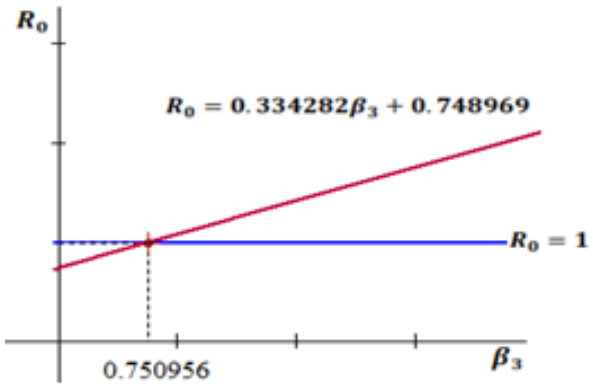


Figure 7.5: Reproduction number versus The horizontal transmission rate of Pre-AIDS.

Figure 7.5: it is graphical representation of the basic reproduction number $R_0$ versus rate of horizontal transmission of the disease from Pre-AIDS class $\beta_3$ and keeping other parameters constant. This figure shows that an increase in the rate of horizontal transmission of Pre-AIDS, makes an increase in the reproduction number. For $\beta_3 > 0.75$ we can see the reproduction number $R_0 > 1$ that indicates the disease persists. When $\beta_3 < 0.75$ the reproduction number $R_0 < 1$ and this indicates that the disease not persists.

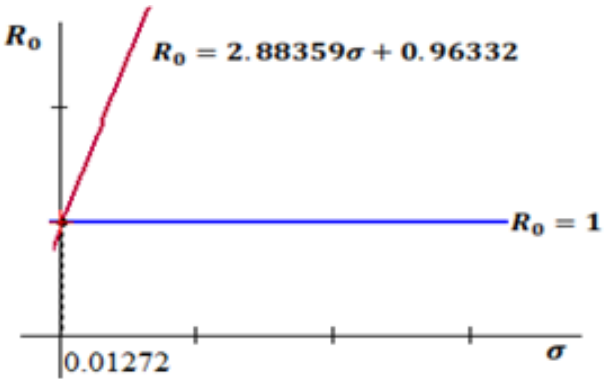


Figure 7.6: Reproduction number versus Blood born transmission.

Figure 7.6: it is graphical representation of the basic reproduction number $R_0$ versus rate of blood born transmission $\sigma$ and keeping other parameters constant. This figure shows that an increase in the rate of blood born transmission between the parametric values 0 and 0.01 makes

an increase in the reproduction number with $R_0 < 1$ and this indicates that the disease not persists. For $\sigma > 0.01$ we can see the reproduction number $R_0 > 1$ that indicates the disease persists.

**Rate of progress of unaware infective to AIDS class and rate of progress of aware infective to AIDS class**

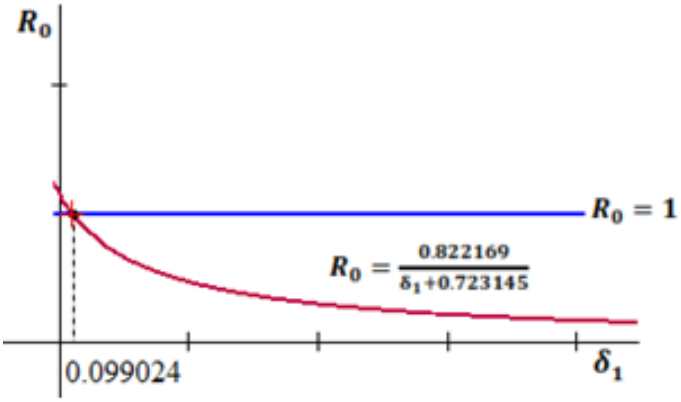


Figure 7.7: Reproduction number versus the rate of progress of unaware infective to AIDS.

Figure 7.7: it is graphical representation of the basic reproduction number $R_0$ versus rate of progress of unaware infective to AIDS, $\delta_1$ and keeping other parameters constant. This figure shows that an increase in the rate of progress of unaware infective to AIDS, between the parametric values 0 and 0.10 makes a decrease in the reproduction number but the reproduction number is greater than one that indicates the disease persists. If the parameter value of $\delta_1$ greater than 0.10, then the reproduction number decreases and becomes less than one where the disease not persists.

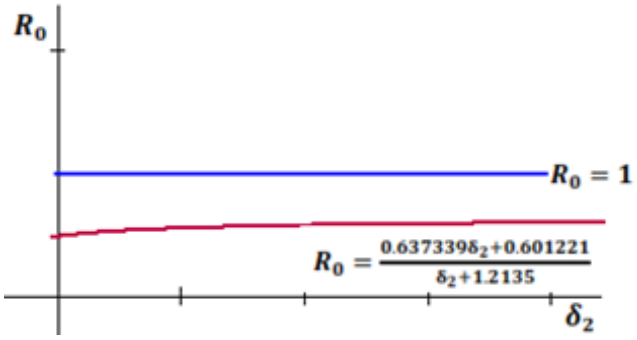


Figure 7.8: Reproduction number versus the rate of progress of aware infective to AIDS.

Figure 7.8: shows that an increase in the rate of progress of aware infective to AIDS, $\delta_2$, the reproduction number, $R_0$ is less than one and has almost constant value. This indicates the

disease not persists.

**Rate of progress of Pre-AIDS to AIDS and rate of transmission of unaware infective to aware infective**

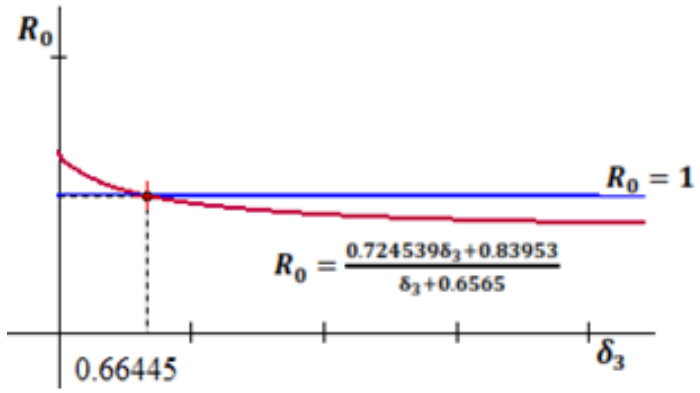


Figure 7.9: Reproduction number versus the rate of progress of Pre-AIDS to AIDS.

Figure 7.9: it is graphical representation of the basic reproduction number $R_0$ versus rate of progress of Pre-AIDS to AIDS, $\delta_3$ and keeping other parameters constant. This figure shows that an increase in the rate of progress of Pre-AIDS to AIDS between the parametric values 0 and 0.66 makes a decrease in the reproduction number but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $\delta_3$ greater than 0.66, then the reproduction number decreases and becomes less than one where the disease not persists.

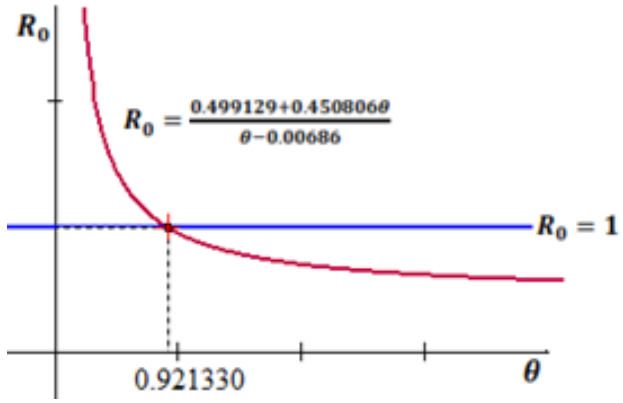


Figure 7.10: Reproduction number versus the rate of transmission of unaware infective to aware infective.

Figure 7.10: it is graphical representation of the basic reproduction number $R_0$ versus rate of transmission of unaware infective to aware infective, $\theta$ and keeping other parameters constant. This figure shows that the rate of transmission of unaware infective to aware infective between

the parametric values 0 and 0.92 makes a decrease in the reproduction number but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $\theta$ greater than 0.92, then the reproduction number decreases and becomes less than one where the disease not persists.

**Rate of transmission of unaware and aware infective to seropositive class**

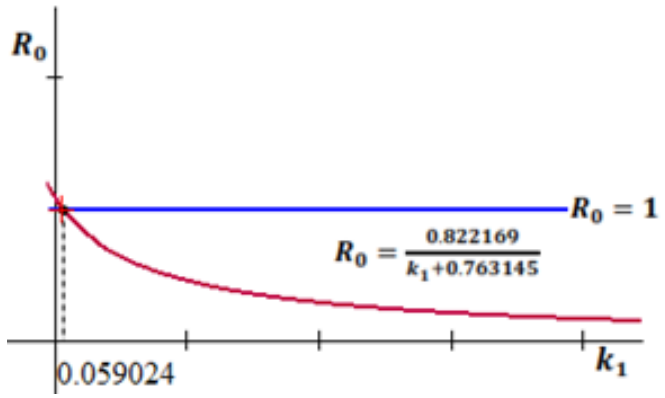


Figure 7.11: Reproduction number versus the rate of transmission of unaware infective to seropositive class.

Figure 7.11: it is graphical representation of the basic reproduction number $R_0$ versus rate of transmission of unaware infective to seropositive class, $k_1$ and keeping other parameters constant. This figure shows that an increase in the rate of transmission of unaware infective to seropositive class between the parametric values 0 and 0.06 makes a decrease in the reproduction number, but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $k_1$ greater than 0.06, then the reproduction number decreases and becomes less than one where the disease dies out.

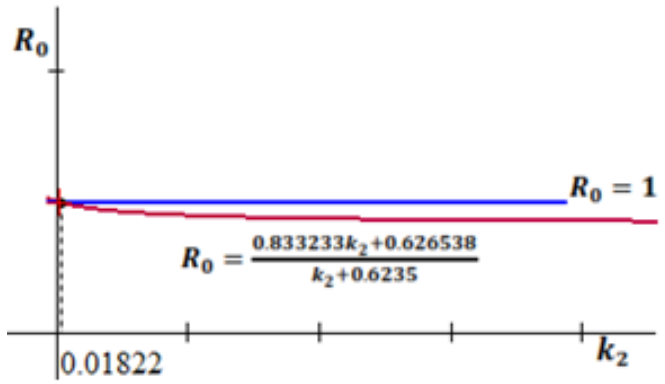


Figure 7.12: Reproduction number versus the rate of transmission of aware infective to seropositive class.

Figure 7.12: it is graphical representation of the basic reproduction number $R_0$ versus rate of

transmission of aware infective to seropositive class, $k_2$ and keeping other parameters constant. This figure shows that an increase in the rate of transmission of aware infective to seropositive class,$k_2$, between the parametric values 0 and 0.02 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $k_2$ greater than 0.02, then the reproduction number is less than one and almost constant in value.

**Rate of transmission of Pre-AIDS individuals to seropositive class and the rate of unaware infective immigrants**

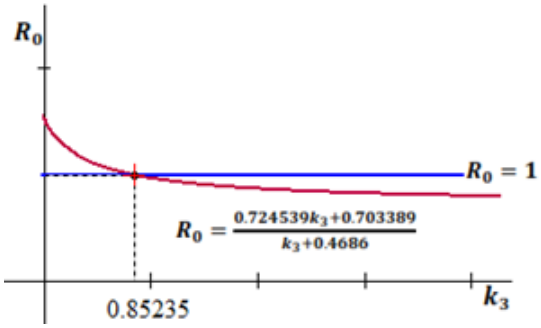


Figure 7.13: Reproduction number versus the rate of transmission of Pre-AIDS to seropositive class.

Figure 7.13:it is graphical representation of the basic reproduction number $R_0$ versus rate of transmission of Pre-AIDS to seropositive class, $k_3$ and keeping other parameters constant. This figure shows that an increase in the rate of transmission of Pre-AIDS to seropositive class between the parametric values 0 and 0.85 makes a decrease in the reproduction number but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $k_3$ greater than 0.85, then the reproduction number is less than one and we can say the disease not persists.

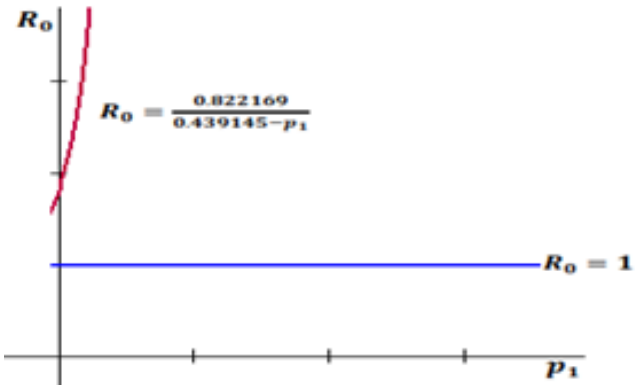


Figure 7.14: Reproduction number versus the rate of unaware infective immigrants.

Figure 7.14:shows that an increase in the rate of unaware infective immigrants,$p_1$, between the parametric values 0 and 0.44 makes an increase in the reproduction number $R_0 > 1$ and tell us the disease persists.

**Rate of aware infective immigrants and rate of progress of unaware infective to Pre-AIDS**

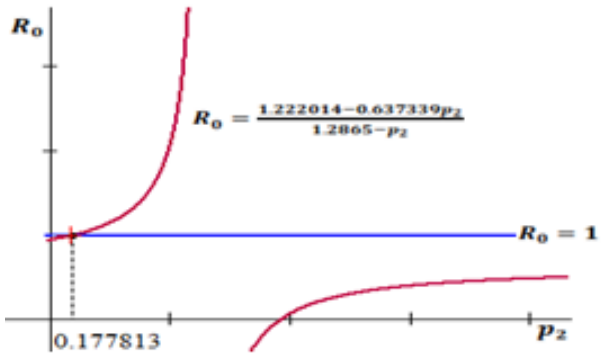


Figure 7.15: Reproduction number versus the rate of aware infective immigrants.

Figure 7.15: it is graphical representation of the basic reproduction number $R_0$ versus rate of transmission of aware infective immigrants, $p_2$ and keeping other parameters constant. This figure shows an increase in the rate of aware infective immigrants between the parametric values 0 and 0.18 makes an increase in the reproduction number but the reproduction number is less than one that indicates the disease not persists. If the rate of aware infective immigrants between 0.18 and 1.29 makes an increase in the reproduction number with, $R_0 > 1$ and tell us the disease persists. Whereas, the rate of aware infective immigrants greater than 1.29, makes an increase in the reproduction number with $R_0 < 1$, and tell us the disease not persists.

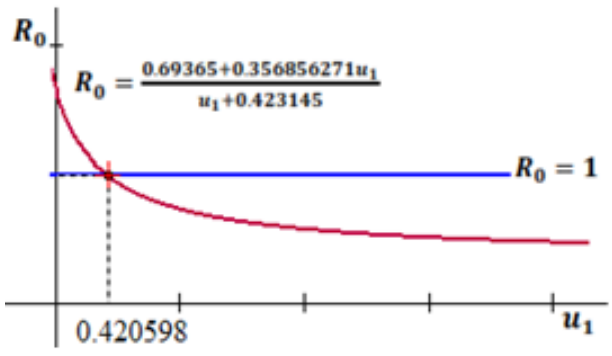


Figure 7.16: Reproduction number versus rate of progress of unaware infective to Pre-AIDS.

Figure 7.16: it is graphical representation of the basic reproduction number $R_0$ versus rate of progress of unaware infective to Pre-AIDS, $u_1$ and keeping other parameters constant. This figure shows an increase in the rate of progress of unaware infective to Pre-AIDS between the

parametric values 0 and 0.42 makes a decrease in the reproduction number but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $u_1$ greater than 0.42, then the reproduction number is less than one and we can say the disease not persists.

**Rate of progress of aware infective to Pre-AIDS and rate of vertical transmission**

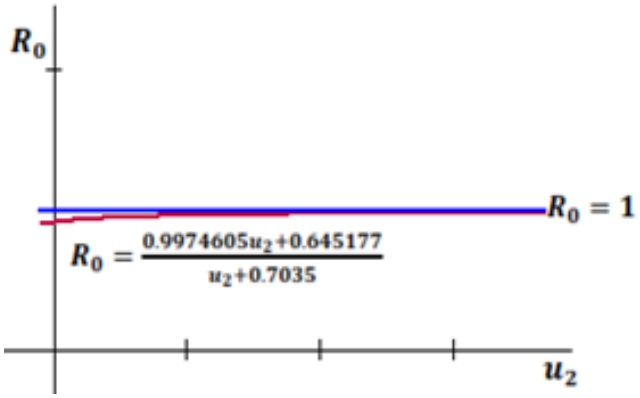


Figure 7.17: Reproduction number versus rate of progress of aware infective to Pre-AIDS.

From figure 7.17, we can observe that an increase in the rate of progress of aware infective to Pre-AIDS, $u_2$, the reproduction number, $R_0$ is less than one and has almost constant value. This indicates the disease not persists.

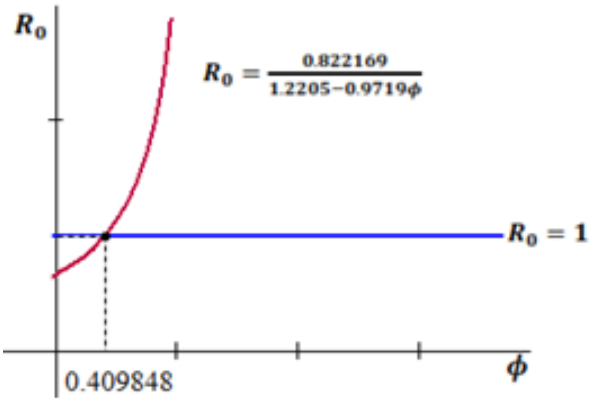


Figure 7.18: Reproduction number versus Rate of vertical transmission.

Figure 7.18: it is graphical representation of the basic reproduction number $R_0$ versus rate of vertical transmission, $\phi$ and keeping other parameters constant. This figure shows that an increase in the rate of vertical transmission between the parametric values 0 and 0.41 makes an increase in the reproduction number but the reproduction number is less than one that indicates the disease not persists. Whereas, the rate of vertical transmission greater than 0.41,

makes an increase in the reproduction number with $R_0 > 1$, and tell us the disease persists.

**The probability of death at birth and natural mortality**

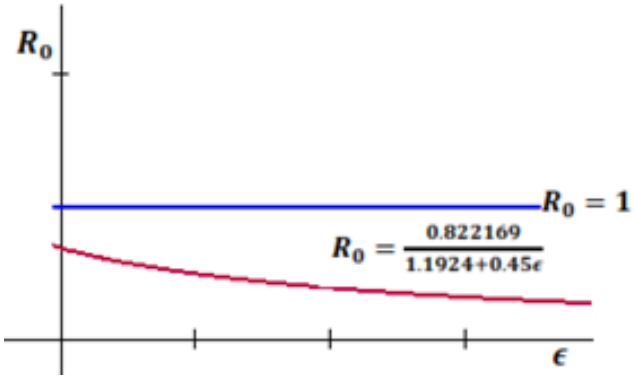


Figure 7.19: Reproduction number versus the probability of death at birth.

Figure 7.19: it is graphical representation of the basic reproduction number $R_0$ versus probability of death at birth, $\epsilon$ and keeping other parameters constant.From this figure we can observe that an increase in the probability of death at birth, $\epsilon$, the reproduction number, $R_0$ is less than one. This indicates the disease not persists.

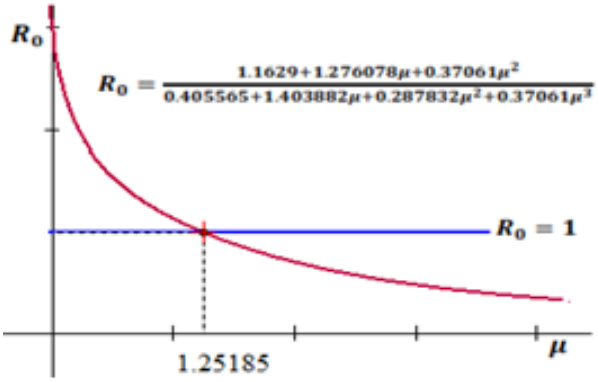


Figure 7.20: Reproduction number versus Natural mortality.

Figure 7.20: it is graphical representation of the basic reproduction number $R_0$ versus natural mortality, $\mu$ and keeping other parameters constant. This figure shows an increase in natural mortality between the parametric values 0 and 1.25 makes a decrease in the reproduction number but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $\mu$ greater than 1.25, then the reproduction number is less than one and we can say the disease not persists.

## 7.2 Conclusions

In this study we have developed a deterministic mathematical model for Age structure and Inflow Immigrants on the Dynamics of HIV/AIDS: dividing susceptible individuals in to sexually immature (i.e age below 15 years) and sexually mature (i.e age 15 years and above), aware and unaware infective, infective immigrants, Pre-AIDS individuals and treatments of infectious individuals. The stability analysis on the model shows that the disease -free equilibrium point $E_0$ is to be locally asymptotically stable and globally asymptotically stable when $R_0 < 1$ and the positive endemic equilibrium point $E^*$ is shown to be locally asymptotically stable and globally asymptotically stable for $Z < Y$. Results from Numerical simulation show that as the transmission rate of unaware infective humans to aware infective increases, the basic reproduction number decreases. This will result in decreasing on the transmission of HIV/AIDS. We evaluated the numerical value of the basic reproduction number. Consequently, $R_0 = 1.05$ that shows the HIV/AIDS disease spread in the community. A sensitivity analysis of the basic reproduction number indicates that the transmission rate of unaware infective humans to aware infective, the rate of vertical transmission and horizontal transmission rate are the most sensitive parameters that can be used to control the spread of the disease.

# Chapter 8

# Results, Conclusions and Recommendations

## 8.1 Results

In this study, we would like to present the results and findings obtained from the analysis of the model discussed under chapter 7.

From sensitive analysis we observed that the most sensitive parameter is the transmission rate of unaware infective humans to aware infective, $\theta$ and the least sensitive parameter is the recruitment into sexually immature class, $Q_2$. The indices having positive signs increase the value of $R_0$ as one increase them and those having negative signs decrease the value of $R_0$, when they are increased.

Results from Numerical simulation show that as the transmission rate of unaware infective humans to aware infective, $\theta$, increases, the basic reproduction number decreases. This will result in decreasing on the transmission of HIV/AIDS.

An increase in the rate of horizontal transmission of unaware infective, $\beta_1$, makes an increase in the reproduction number, $R_0$. If $\beta_1 > 0.73$ the reproduction number $R_0 > 1$ that indicates the disease persists. When $\beta_1 < 0.73$ the reproduction number $R_0 < 1$ this indicates that the disease not persists.

From figure 7.4 we can observe that an increase in the rate of horizontal transmission of aware infective, $\beta_2$, makes an increase in the reproduction number, $R_0$. If $\beta_2 > 0.55$ the reproduction

number $R_0 > 1$ that indicates the disease persists. When $\beta_2 < 0.55$ the reproduction number $R_0 < 1$ this indicates that the disease not persists.

We can observe From figure 7.5, that an increase in the rate of horizontal transmission of Pre-AIDS, $\beta_3$, makes an increase in the reproduction number, $R_0$. For $\beta_3 > 0.75$ we can see the reproduction number $R_0 > 1$ that indicates the disease persists. When $\beta_3 < 0.75$ the reproduction number $R_0 < 1$ and this indicates that the disease not persists.

We can observe that an increase in the rate of progress of unaware infective to AIDS, $\delta_1$, between the parametric values 0 and 0.10 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists. If the parameter value of $\delta_1$ greater than 0.10, then the reproduction number decreases and becomes less than one where the disease not persists.

From figure 7.9, we can observe that an increase in the rate of progress of Pre-AIDS to AIDS, $\delta_3$, between the parametric values 0 and 0.66 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $\delta_3$ greater than 0.66, then the reproduction number decreases and becomes less than one where the disease not persists.

An increase in the rate of transmission of unaware infective to aware infective, $\theta$, between the parametric values 0 and 0.92 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $\theta$ greater than 0.92, then the reproduction number decreases and becomes less than one where the disease not persists.

Figure 7.11 shows an increase in the rate of transmission of unaware infective to seropositive class, $k_1$, between the parametric values 0 and 0.06 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $k_1$ greater than 0.06, then the reproduction number decreases and becomes less than one where the disease not persists.

From figure 7.12 we can observe that an increase in the rate of transmission of aware infective to seropositive class, $k_2$, between the parametric values 0 and 0.02 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $k_2$ greater than 0.02, then the reproduction number is less than one and almost constant in value.

An increase in the rate of transmission of Pre-AIDS to seropositive class, $k_3$, between the parametric values 0 and 0.85 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the

parameter value of $k_3$ greater than 0.85, then the reproduction number is less than one and we can say the disease not persists.

Figure 7.14 shows that an increase in the rate of unaware infective immigrants, $p_1$, between the parametric values 0 and 0.44 makes an increase in the reproduction number $R_0 > 1$ and tell us the disease persists.

From figure 7.15, we can see that an increase in the rate of aware infective immigrants, $p_2$, between the parametric values 0 and 0.18 makes an increase in the reproduction number, $R_0$, but the reproduction number is less than one that indicates the disease not persists. If the rate of aware infective immigrants between 0.18 and 1.29 makes an increase in the reproduction number with, $R_0 > 1$ and tell us the disease persists. Whereas, the rate of aware infective immigrants greater than 1.29, makes an increase in the reproduction number, $R_0 < 1$, and tell us the disease not persists.

An increase in the rate of progress of unaware infective to Pre-AIDS, $u_1$, between the parametric values 0 and 0.42 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $u_1$ greater than 0.42, then the reproduction number is less than one and we can say the disease not persists.

From figure 7.17, we can observe that an increase in the rate of progress of aware infective to Pre-AIDS, $u_2$, the reproduction number, $R_0$ is less than one and has almost constant value. This indicates the disease not persists.

Figure 7.18 shows an increase in the rate of vertical transmission, $\phi$, between the parametric values 0 and 0.41 makes an increase in the reproduction number, $R_0$, but the reproduction number is less than one that indicates the disease not persists. Whereas, the rate of vertical transmission greater than 0.41, makes an increase in the reproduction number, $R_0 > 1$, and tell us the disease persists.

From figure 7.19, we can observe that an increase in the probability of death at birth, $\epsilon$, the reproduction number, $R_0$ is less than one and has almost constant value. This indicates the disease not persists.

An increase in natural mortality, $\mu$, between the parametric values 0 and 1.25 makes a decrease in the reproduction number, $R_0$, but the reproduction number is greater than one that indicates the disease persists with a decreasing rate. If the parameter value of $\mu$ greater than 1.25, then the reproduction number is less than one and we can say the disease not persists.

## 8.2 Conclusions

In this thesis, we studied a mathematical model analysis of the dynamics of HIV/AIDS with different mode of transmission and inflow immigrants in Ethiopia. Persistence of current infections and their possible dynamics are investigated. We also addressed the stability of the epidemic, sensitivity analysis and numerical simulation.

In Chapters 1-3, we reviewed some basic features of the HIV epidemic, historical background, and the different mathematical models investigated by different authors. And also highlights the role of modeling and stated the method we used in the study.

In Chapter 4, we proposed an improvement of the model [104] that is to show the effect of unaware infective immigrants, aware infective immigrants, vertical and blood borne transmissions and treatment on the dynamics of HIV/AIDS. A non-linear differential equation was formulated to represent the model. The stability analysis on the model was investigated.

In Chapter 5, we extended the model given in chapter 4 and studied. Here we have developed a deterministic mathematical model for Age structure and Inflow Immigrants on the Dynamics of HIV/AIDS: dividing susceptible individuals in to sexually immature (i.e age below 15 years) and sexually mature (i.e age 15 years and above), aware and unaware infective, infective immigrants, Pre-AIDS individuals and treatments of infectious individuals. The stability analysis of the model also analyzed.

In Chapter 6, we investigated the model proposed in chapter 4 using parameter values obtained from different journals. A sensitivity analysis of the basic reproduction number indicates that transmission probability, the rate of progress to AIDS and the rate of aware infective immigrants are the most sensitive parameters that can be used to control the spread of the disease. Results from numerical simulation show that as the probability of transmission of the disease to susceptible individuals by unaware and aware infective individuals increases, the basic reproduction number also increases. This will result in increasing on the transmission of HIV/AIDS.

In Chapter 7, we analyzed the model given in chapter 5 using parameter values obtained from data taken from Ethiopia and related materials. Results from numerical simulation show that as the transmission rate of unaware infective humans to aware infective increases, the basic reproduction number decreases. This will result in decreasing on the transmission of HIV/AIDS. We evaluated the numerical value of the basic reproduction number. Consequently, $R_0 = 1.05$ that shows the HIV/AIDS disease spread in the community. A sensitivity analysis of the basic reproduction number indicates that the transmission rate of unaware infective humans to

aware infective, the rate of vertical transmission and horizontal transmission rate are the most sensitive parameters that can be used to control the spread of the disease.

As further studies and future directions, one may include additional realistic features in our models. Some additional aspects include: age structure in infectious stages, sex structured, investigate time dependency on transmission rate and other parameters of a model under a continuous changes of control measures. Drug resistance is a critical issue in HIV infection, thus by including drug resistance in our models one can improve modeling outcomes significantly.

## 8.3 Recommendations

Based on the above results and discussion we observed the basic reproduction number $R_0 = 1.05$ is greater than one and this implies that the disease spreads in the community. In order to decrease the spread of HIV/AIDS in the society, we recommend the following based on the most influential parameters.

The first control parameter is the rate of transmission of unaware infective to aware infective $\theta$. $\theta = \frac{\text{number of unaware infected who know their status per year}}{\text{total number of unaware infective}} = 0.79$, where number of population moving from unaware infected to aware infected class is 64,104 and total number of unaware infective is 81,144. The intersection point of $R_0 = 1$ and the rate of transmission of unaware infective to aware infective class $\theta$ is $(\theta, R_0) = (0.92, 1)$. Therefore, for basic reproduction to be less than unity, the control parameter $\theta$ should be greater than 0.92. But from the real data we obtained that $\theta = \frac{64{,}104}{81{,}144} = 0.79$. Hence, this value should approach 0.92 by fixing the total number of unaware infected population 81,144 and increase the number of population moving from unaware infected to aware infected class from 64,104 to 74,652.

The second control parameter is the rate of vertical transmission $\phi$.
$\phi = \frac{\text{Average number of infected new born per year}}{\text{Total number of new born}} = 0.45$, where average number of infected new born per year is 148,709 and total number of new born is 330,465. The intersection point of $R_0 = 1$ and the rate of vertical transmission $\phi$ is $(\phi, R_0) = (0.41, 1)$. Therefore, for basic reproduction to be less than unity, the control parameter $\phi$ should be less than 0.41. But from the real data we obtained that $\phi = \frac{148{,}709}{330{,}465} = 0.45$. Hence, this value should approach 0.41 by decreasing infected

new born as much as possible.

The third control parameter is the horizontal transmission rate of unaware infective to susceptible individuals $\beta_1$. $\beta_1 = \frac{\text{Effective contact of unaware infective}}{\text{Total contact of unaware infective}} = 0.83$, where effective contact of unaware infective is 67,350 and total contact of unaware infective is 81,144. The intersection point of $R_0 = 1$ and the horizontal transmission rate of unaware infective to susceptible individuals $\beta_1$ is $(\beta_1, R_0) = (0.73, 1)$. Therefore, for basic reproduction to be less than unity, the control parameter $\beta_1$ should be less than 0.73. But from the real data we obtained that$\beta_1 = \frac{67{,}350}{81{,}144} = 0.83$. Hence, this value should approach 0.73 by fixing the total contact of unaware infective 81,144 and decrease effective contact of unaware infective from 67,350 to 59,235.

The fourth control parameter is the rate of progress of unaware infective to Pre-AIDS $u_1$. $u_1 = \frac{1}{\text{Average life time of unaware infective individual progress to Pre-AIDS}} = 0.36$, where average life time of unaware infective individual progress to Pre-AIDS is 2.78. The intersection point of $R_0 = 1$ and the rate of progress of unaware infective to Pre-AIDS $u_1$ is $(u_1, R_0) = (0.42, 1)$. Therefore, for basic reproduction to be less than unity, the control parameter $u_1$ should be greater than 0.42. But from the real data we obtained that $u_1 = \frac{1}{2.78} = 0.36$. Hence, this value should approach 0.42 by decreasing average life time of unaware infective individual progress to Pre-AIDS from 2.78 to 2.38.

The fifth control parameter is the horizontal transmission rate of Pre-AIDS to susceptible individuals $\beta_3$. $\beta_3 = \frac{\text{Effective contact of Pre-AIDS}}{\text{Total contact of Pre-AIDS individuals}} = 0.9$, where effective contact of Pre-AIDS is 27,265 and total contact of Pre-AIDS individuals is 30,294. The intersection point of $R_0 = 1$ and the horizontal transmission rate of Pre-AIDS to susceptible individuals $\beta_3$ is $(\beta_3, R_0) = (0.75, 1)$. Therefore, for basic reproduction to be less than unity, the control parameter $\beta_3$ should be less than 0.75. But from the real data we obtained that $\beta_3 = \frac{27{,}265}{30{,}294} = 0.9$. Hence, this value should approach 0.75 by fixing the total contact of Pre-AIDS individuals 30,294 and decrease effective contact of Pre-AIDS individuals from 27,265 to 22,721.

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