DEBRE BERHAN UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE
DEPARTMENT OF MATHEMATICS

APPLICATION TO DIFFERENTIAL TRANSFORM METHOD FOR SOLVING INITIAL VALUE PROBLEM OF ORDINARY DIFFERENTIAL EQUATIONS

A PROJECT ON THE PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE IN DIFFERENTIAL EQUATION

BY

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DECLARATION

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any degree, diploma, associate ship, fellowship or any other similar title to me.

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This is to officially state that dissertation entitled by **APPLICATION TO DIFFERENTIAL TRANSFORM METHOD FOR SOLVING INITIAL VALUE PROBLEM OF ORDINARY DIFFERENTIAL EQUATIONS** is the original work done by Mr. MULUGETA ANDUALEM ABATE ID.NO DBUPGR/013/08.
This paper has done by Mr. MULUGETA ANDUALEM ABATE for the partial fulfillment of the degree of Master of Science in Differential Equation from Debre Berhan University.
Therefore, I recommend that it would be accepted as fulfilling the project requirement.

Prof.A.N.Mohamad

Main Advisor

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ABBREVIATIONS

BVPs   Boundary-value problems

IVPs   Initial-value problems

DEs   Differential equations

$\delta_{i,j}$ The Kronecker Delta function

PDE   Partial differential equation

DTM   Differential transform method

ODE   Ordinary differential equation

LIST OF FIGURES

Abstract

In this paper, we present approximate and exact solution of initial value problems of ordinary differential equations (Odes) by using Differential Transform method (DTM). The concept of differential transform and some properties of differential transform also presented.
Various examples of ordinary differential equations solved analytically and numerically, in order to test the ability and accuracy of differential transform method we compare the numerical and exact solution in graphically.

Chapter 1

Introduction

1.1 Background of the study

Ordinary or partial differential equations(PDEs) are commonly encountered in several branches of sciences including Biology, Physics, Chemistry and Mathematics [12,15]. A variety of methods, exact, approximate and purely numerical are available for the solution of differential equations. Most of these methods are computationally intensive because they are trial-and error in nature, or need complicated symbolic computations. The differential transformation technique is one of the numerical methods for ordinary differential equations.
The concept of differential transformation was first proposed by Zhou in 1986 [6,7,8,19] and it was applied to solve linear and non-linear initial value problems in electric circuit analysis.

This method constructs a semi analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming especially for high order equations. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The Differential transformation method is very effective and powerful for solving various kinds of differential equation.

For example, it was applied to two point boundary value problems [8], to differential-algebraic equations [4], to the KdV and mKdV equations [11], to the Schrodinger equations [13] to linear differential equations [9] to fractional differential equations [3] and to the Riccati differential equation [5]. Jang et al. [10] introduced the application of the concept of the differential transformation of fixed grid size to approximate solutions of linear and non-linear initial value problems. Abdel Hassan [1] applied the differential transformation technique of fixed grid size to solve the higher-order initial value problems. The transformation method can be used to evaluate the approximating solution by the finite Taylor series and by an iteration procedure described by the transformed equations obtained from the original equation using the operations of differential transformation. The main advantage of this method is that it can be applied directly to nonlinear ODEs as a linear.
Another important advantage is that this method is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate and also does not generate secular terms (noise terms) and does not need to analytical integration as other semi-analytical numerical methods [16].

In this paper, the differential transformation technique is applied to solve initial value problems of ordinary differential equation (variable coefficient, constant coefficient, homogeneous and non homogeneous). The method can be used to evaluate the approximating solution by an iteration procedure described by the transformed equation obtained from the original equation using the operations of differential transformation. The differential transformation technique can be used to obtain both the numerical and analytical solutions of both linear and nonlinear differential equations. The organization of this paper is as follows:

Section (1) describes the DTM and fundamental theorems of differential transform method in order to solve initial value problems.
In section (2), some analytical and numerical examples are presented to illustrate the efficiency of the DTM and obtained numerical results are compared to the exact solution in graphically.
Finally, we give the conclusion.

1.2 Statement Of The Problem

Ordinary differential equation arises in different area of applied sciences such as Engineering, Physics and Chemistry.

But finding the solution of such problems are not easy specially nonlinear differential problems (non linearity of logarithmic, non linearity exponential) because of the difficulties that are caused by the nonlinear terms.

There are several analytic and numerical methods of solving initial value problems of ordinary differential equation. One of those methods is Taylor series method.

However, the traditional Taylor series method requires the calculation of higher order derivatives, a difficult symbolic, and it takes time.

this project focused on the Differential Transformation method (DTM) applied to solve initial value problems of ordinary differential equations (variable coefficient, constant coefficient, homogeneous and non homogeneous), since differential transform method is simple and easy to use and produce reliable results. The method also minimize the computational difficulties of the Taylor series method

1.3 Objectives

1.3.1 General Objective

- The main objective of this project is solving initial value problems of ordinary differential equations by Differential Transform.

1.3.2 Specific Objectives

1. To describe the fundamental theorem with proof of Differential Transform method .

2. To demonstrate different examples of initial value problems of ordinary differential equations and show its applicability for this kind of equation.

3. To describe the basic definition of Differential Transform method.

4. To compare the exact solution with DTM to test the efficiency and the accuracy of DTM.

1.3.3 Significance of the study

This study will be conducted basically on solution of initial value problems of ordinary differential equations.

The purpose of this paper will be understand how to solve initial value problems of ordinary differential equations with the help of differential transform method.

- The main advantage of the method is that, it can be applied directly to various types of differential and integral equations, which are linear and nonlinear, homogeneous and non-homogeneous, with constant and with variable coefficients.

- Differential transform method is easy to handle and compute the works in less time even when apply nonlinear differential equation.

1.4 Preliminary

1.4.1 The Kronecker Delta

Definition 1.4.1.1: The Kronecker Delta $\delta_{i,j}$ is a function of two argument i and j.
If i and j are the same value (i.e $i = j$) then the function $\delta_{i,j} = 1$, other wise the Kronecker Delta is equal to zero.

Formally this is written by, $\delta_{i,j} = \begin{cases} 1 & \text{if } i = j. \\ 0 & \text{if } i \neq j. \end{cases}$

Caution: Of course that the variables i and j don't always have to be specifically the letter i and j. They could be m and n letters or the authors what they like.
Furthermore, some authors prefer to leave out the comma entirely.

$$\text{i.e} \quad \delta_{i,j} = \delta_{ij}.$$

1.4.2 Differential Equation

Definition 1.4.2.1: A differential equation is an equation containing the derivatives of one or more dependent variable, with respect to one or more independent variable. In general, the unknown function may depend on several variables and the equation may include various partial derivatives.
However, in this paper we consider only the differential equation for a function of a single real variable such equations are an ordinary differential equations and may be classified as either initial value problems (IVPs) or boundary value problems (BVPs).
As we know, most differential equations have more than one solution. For a first order differential equation, the general solution usually involves an arbitrary constant c with one particular solution corresponding to each value of c.
What this means is that knowing a differential equation that a function $y(x)$ satisfies is not enough information to determine $y(x)$.

Definition 1.4.2.2: Initial value problem The problem of finding a function satisfying a differential equation and an initial condition is an initial value problem (IVP). In other word a differential equation together with some specific conditions on the dependent variable and its derivatives which are given at the same value of the independent value is Initial Value Problems. The specific conditions are said to be initial conditions.
On the other hand, if the specific conditions are given at different values of the independent variable, the problems known as Boundary Value Problem and the specific conditions are said to be boundary conditions.

1.4.3 Initial-Value problem

Example for n-th order initial-value problem : An n-th order initial-value problem consists of n-th order differential equation

$$F(x, y, y', y'', ...y^{(n)})$$

Together with n (initial) conditions of the form

$$y(x_0) = c_0, y'(x_0) = c_1, y''(x_0) = c_2, ..., y^{(n-1)}(x_0) = c_n.$$

at the same value of $x = x_0$, where F is a given function of $n+2$ variables and $y = y(x)$ is unknown function of a real variable x.
The maximal order of the derivative $y^{(n)}$ in the above ordinary differential equation called the order of the ODE.
Where x_0 and $c_0, c_1, c_2, ..., c_{n-1}$ are given numbers.
Example :

$$y'' + 9y = 0$$
$$y(0) = 0,\ y(\pi) = 0$$

Is not initial value problem; the two conditions are not the form in the definition, namely $y(x_0) = \alpha, y'(x_0) = \beta$.
Example for n-th order boundary value problem :

$$F(x, y, y', y'', ...y^{(n)})$$

Together with n (initial) conditions of the form

$$y(x_0) = c_0,\ y'(x_0) = c_1,\ y''(x_1) = c_2, ..., y^{(n-1)}(x_n) = c_n$$

At different value of $x = x_0, x = x_1, x = x_2, ..., x = x_n$

Definition 1.4.3.1: A differential equation which is given by in the form
$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + a_{n-2}(x)y^{(n-2)} + a_1(x)y' + a_0(x)y = q(x)$ is a linear DE of order n. The above DE is linear since function y and all derivatives occur only in the first power and there are no products of $y, y', ...y^{(n)}$ i.e it is linear in the dependent variable. The function $a_i(x)$ for $i = 1, 2, ..., n$ depend on variable x and can be arbitrary, other wise nonlinear differential equation.

Definition 1.4.3.2: If $q(x) = 0$ in the above equation then the differential equation is homogeneous (differential equation is homogeneous if it has no terms that are functions of the independent variable alone), otherwise i.e $q(x) \neq 0$, i.e an equation in which there are terms that are functions of the independent variables alone then the differential equation is non-homogeneous

1.5 Differential transform

In this section, we introduce the concept of one dimensional differential transform and some basic fundamental theorems .

The basic definition of one dimensional differential transform of the k^{th} derivatives of a function $f(x)$ is defined as follow.
An arbitrary function $f(x)$ can be expanded in Taylor series about $x = a$ as:

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} \left[\frac{d^k}{dx^k} f(x) \right] \Big|_{x=a} \tag{1.1}$$

The particular case of Eq.(1.1) when $a = 0$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k}{dx^k} f(x) \right] \Big|_{x=0} \tag{1.2}$$

Now, the differential transform is given by

$$F(k) = \frac{1}{k!} \left[\frac{d^k}{dx^k} f(x) \right] \Big|_{x=0} \tag{1.3}$$

Where $f(x)$ is the original function and $F(k)$ is the transformed function and k is a set of non negative integer and $\frac{d^k}{dx^k}$ means the k^{th} derivative with respect to x. In this paper the lower case and upper case letters represent the original and transformed function respectively.
The Differential inverse transform of $F(k)$ is defined as:

$$f(x) = \sum_{k=0}^{\infty} (x^k) F(k) \tag{1.4}$$

Eq.(1.4) can be obtain as follow:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k}{dx^k} f(x) \right] \Big|_{x=0}. \tag{1.5}$$

Then expand it, we get

$$f(x) = \frac{x^0}{0!} \left[\frac{d^0}{dx^0} f(x) \right] \Big|_{x=0} + \frac{x^1}{1!} \left[\frac{d^1}{dx^1} f(x) \right] \Big|_{x=0} + \frac{x^2}{2!} \left[\frac{d^2}{dx^2} f(x) \right] \Big|_{x=0} \cdots \tag{1.6}$$

From Eq.(1.3), we get

$$\begin{aligned}
F(0) &= \frac{1}{0!} [\frac{d^0}{dx^0} f(x)] \Big|_{x=0} \\
F(1) &= \frac{1}{1!} [\frac{d^1}{dx^1} f(x)] \Big|_{x=0} \\
F(2) &= \frac{1}{2!} [\frac{d^2}{dx^2} f(x)] \Big|_{x=0} \\
F(3) &= \frac{1}{3!} [\frac{d^3}{dx^3} f(x)] \Big|_{x=0}
\end{aligned}$$

$$\vdots$$

Since

$$\begin{aligned}
F(k) &= \frac{1}{k!}[\frac{d^k}{dx^k}f(x)]\Big|_{x=0} \\
\Rightarrow f(x) &= x^0F(0) + x^1F(1) + x^2F(2) + ... \\
f(x) &= \sum_{k=0}^{\infty}(x^k)F(k)
\end{aligned}$$

It is clear that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate symbolically. However, relative derivatives are calculated by an iterative ways which are described by the transformed equations of the original functions. Differential transform method is easy to handle and compute the works in less time even when apply nonlinear differential equation.
From equation (1.2) and (1.3) the following fundamental theorem can be proved.

Theorem 1 : *If $f(x) = h(x) + g(x)$, then $F(k) = H(k) + G(k)$*

Proof : By using the definition of differential transform

$$\begin{aligned}
H(k) &= \frac{1}{k!}\tfrac{d^k}{dx^k}h(x)\Big|_{x=0} \\
h(x) &= \sum_{k=0}^{\infty}(x^k)H(k) \\
G(k) &= \frac{1}{(k)!}\tfrac{d^k}{dx^k}g(x)\Big|_{x=0} \\
g(x) &= \sum_{k=0}^{\infty}(x^k)G(k)
\end{aligned}$$

Now,

$$\begin{aligned}
H(k) + G(k) &= \frac{1}{(k)!}\tfrac{d^k}{dx^k}h(x) + \frac{1}{(k)!}\tfrac{d^k}{dx^k}g(x) \\
&= \frac{1}{(k)!}\tfrac{d^k}{dx^k}[h(x) + g(x)]
\end{aligned}$$

But from the hypothesis

$$\begin{aligned}
f(x) &= h(x) + g(x) \\
\Rightarrow H(k) + G(k) &= \frac{1}{(k)!}\tfrac{d^k}{dx^k}[f(x)] \\
\Rightarrow H(k) + G(k) &= F(k) \qquad \square
\end{aligned}$$

Theorem 2 : *If $f(x) = ch(x)$, then $F(k) = cH(k)$, where c, it is a constant*

Proof : By using the definition of differential transform

$$\begin{aligned} h(x) &= \sum_{k=0}^{\infty}(x^k)H(k) \\ F(k) &= c\Big[\frac{1}{(k)!}\tfrac{d^k}{dx^k}h(x)\Big] \\ &= cH(k) \end{aligned}$$

□

Theorem 3 : *If* $f(x) = \frac{d}{dx}h(x)$, *then* $F(k) = (k+1)H(k+1)$

Proof : By using the definition of differential transform

$$\begin{aligned} h(x) &= \sum_{k=0}^{\infty}(x^k)H(k) \\ &= H(0) + xH(1) + x^2H(2) + x^3H(3) + ... \end{aligned}$$

Differentiate both sides with respect to x.

$$\begin{aligned} \Rightarrow \tfrac{d}{dx}h(x) &= H(1) + 2xH(2) + 3x^2H(3) + ... \\ &= \sum_{k=0}^{\infty}(x^k)(k+1)H(k+1) \\ \Rightarrow f(x) &= \tfrac{d}{dx}h(x) \\ &= \sum_{k=0}^{\infty}(x^k)(k+1)H(k+1) \end{aligned}$$

Consequently we obtain $F(k) = (k+1)H(k+1)$ □

Theorem 4 : *If* $f(x) = \frac{d^n}{dx^n}h(x)$, *then* $F(k) = \frac{(k+n)!}{k!}H(k+n)$

Proof : By using the definition of differential transform

$$\begin{aligned} h(x) &= \sum_{k=0}^{\infty}(x^k)H(k) \\ &= H(0) + xH(1) + x^2H(2) + x^3H(3) + ... \end{aligned}$$

Differentiate both sides with respect to x

$$\begin{aligned} \Rightarrow \tfrac{d}{dx}h(x) &= H(1) + 2xH(2) + 3x^2H(3) + ... \\ \tfrac{d}{dx}h(x) &= \sum_{k=0}^{\infty}(x^k)(k+1)H(k+1) \\ \tfrac{d^2}{dx^2}h(x) &= 2H(2) + 6xH(3) + ... \\ \tfrac{d^2}{dx^2}h(x) &= \sum_{k=0}^{\infty}(x^k)(k+1)(k+2)H(k+2) \end{aligned}$$

$$\vdots$$

$$\begin{aligned} f(x) &= \tfrac{d^n}{dx^n}h(x) \\ &= \sum_{k=0}^{\infty}(x^k)(k+1)(k+2)...(k+n)H(k+n) \end{aligned}$$

We have

$$\begin{aligned} F(k) &= (k+1)(k+2)...(k+n)H(k+n) \\ &= \frac{(k+n)!}{k!}H(k+n) \end{aligned} \qquad \square$$

Theorem 5 : *If* $f(x) = h(x)g(x)$, *then* $F(k) = \sum_{k=0}^{\infty} H(m)G(k-m)$

Proof :By using the definition of differential transform

$$f(x) = \sum_{m=0}^{\infty} x^m H(m) \sum_{j=0}^{\infty} x^j G(j)$$

$$f(x) = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{k} H(m)G(k-m)$$

$$F(k) = \sum_{m=0}^{k} H(m)G(k-m) \qquad \square$$

Theorem 6 : *Let* $f(x)$, *be an analytic function, with* $D_T\{f(x)\} = F(k)$, *then* $D_T\{e^{ax}f^{(n)}(x)\} = \sum_{i=0}^{k} \frac{a^k}{k!}\frac{(k+n-i)!}{(k-i)!}F(k+n-i)$

Proof : Using the definition of differential transform let

$$\begin{aligned} F_1(k) &= D_T\{e^{ax}\} = \frac{a^k}{k!} \\ F_2(k) &= D_T\{f^{(n)}(x)\} = \frac{(k+n)!}{k!}F(k+n) \end{aligned}$$

Then by using theorem (5)

$$D_T\{e^{ax}f^{(n)}(x)\} = \sum_{i=0}^{k} F_1(i)F_2(k-i)$$

$$\Rightarrow D_T\{e^{ax}f^{(n)}(x)\} = \sum_{i=0}^{k} \frac{a^i}{i!}\frac{(k+n-i)!}{(k-i)!}F(k+n-i) \qquad \square$$

Theorem 7 : *If* $f(x) = c$ *then,* $F(k) = c\delta(k)$
Where c *is a constant and* $\delta(k)$ *is Kronecker delta*

Proof : By using the definition of differential transform

$$f(x) = \sum_{k=0}^{\infty} x^k F(k)$$

$$\text{Since } f(x) = c$$

$$\Rightarrow c = F(0) + xF(1) + x^2F(2) + x^3F(3) + ...$$

From the definition of the polynomials.

$$F(0) = c$$

$$F(1) = F(2) = ... = 0$$

$$\Rightarrow F(k) = \begin{cases} c & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$\Rightarrow F(k) = c \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$\Rightarrow F(k) = c\delta(k).$$

□

Theorem 8 : *If* $f(x) = x$*, then* $F(k) = \delta(k-1)$

Proof :By using the definition of differential transform

$$f(x) = \sum_{k=0}^{\infty} x^k F(k)$$

$$\text{Since } f(x) = x$$

$$\Rightarrow x = F(0) + xF(1) + x^2F(2) + x^3F(3) + ...$$

From the definition of the polynomials

$$F(1) = 1$$

$$F(0) = F(2) = ... = 0$$

$$\Rightarrow F(k) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{cases}$$

$$\Rightarrow F(k) = \begin{cases} 1 & \text{if } k-1 = 0 \\ 0 & \text{if } k-1 \neq 0 \end{cases}$$

$$\Rightarrow F(k) = \delta(k-1)$$

□

Theorem 9 : *If* $f(x) = x^m$*, then* $F(k) = \delta(k-m)$

Proof : By using the definition of differential transform

$$f(x) = \sum_{k=0}^{\infty} x^k F(k)$$

$$\text{Since} \qquad f(x) = x^m$$

$$\Rightarrow x^m = F(0) + xF(1) + x^2F(2) + x^3F(3) + ... + x^mF(m) + ...$$

From the definition of the polynomials.

$$F(0) = F(1) = F(2) = ...F(m-1) = 0 \; and \; F(m) = 1$$

$$\Rightarrow F(k) = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$$

$$\Rightarrow F(k) = \begin{cases} 1 & \text{if } k - m = 0 \\ 0 & \text{if } k - m \neq 0 \end{cases}$$

$$\Rightarrow F(k) = \delta(k-m) \qquad \square$$

Theorem 10 *: Let $f(x)$, be an analytic function, with $D_T\{f(x)\} = F(k)$, then $D_T\{x^m f^{(n)}(x)\} = \sum_{i=0}^{k} \delta_{im} \frac{(k+n-i)!}{(k-i)!} F(k+n-i)$*

Proof : By using theorem 5

$$\text{Since, } D_T\{x^m\} = \delta_{km}$$

$$\Rightarrow D_T\{x^m\} = F_1(k) = \delta_{km}$$

$$F_2(k) = D_T\{f^{(n)}(x)\} = \frac{(k+n)!}{k!} F(k+n$$

$$\Rightarrow D_T\{x^m f^{(n)}(x)\} = \sum_{i=0}^{k} F_1(i) F_2(k-i)$$

$$\text{Therefore, } D_T\{x^m f^{(n)}(x)\} = \sum_{i=0}^{k} \delta_{im} \frac{(k+n-i)!}{(k-i)!} F(k+n-i). \qquad \square$$

Theorem 11 *: If $f(x) = e^{\lambda x}$, then $F(k) = \frac{\lambda^k}{k!}$*

Proof : By using the definition of differential transform

$$f(x) = \sum_{k=0}^{\infty} x^k F(k)$$

We use Taylor series expansion of $e^{\lambda x}$
$1 + \lambda \times x + \frac{\lambda^2}{2!} x^2 + \frac{\lambda^3}{3!} x^3 + ... = F(0) + xF(1) + x^2 F(2) + x^3 F(3) + ...$

$$F(0) = 1$$

$$F(1) = \lambda$$

$$F(2) = \frac{\lambda^2}{2!}$$

$$(3) = \frac{\lambda^3}{3!}$$

$$\vdots$$

$$F(k) = \frac{\lambda^k}{k!} \qquad \square$$

Theorem 12 : *If* $f(x) = (1+x)^m$, *then* $F(k) = \frac{m(m-1)...(m-k+1)}{k!}$

Proof : By using the definition of differential transform

$$f(x) = \sum_{k=0}^{\infty} x^k F(k)$$

We use Binomial theorem of $(1+x)^m$

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + ... = F(0) + xF(1) + x^2F(2) + ...$$

$$\begin{aligned} \text{Then, } \; F(0) &= 1 \\ F(1) &= m \\ F(2) &= \frac{m(m-1)}{2!} \\ F(3) &= \frac{m(m-1)(m-2)}{3!} \end{aligned}$$

$$\vdots$$

$$F(k) = \frac{m(m-1)(m-2)...(m-(k-1))}{k!} \qquad \square$$

Theorem 13 : *If* $f(x) = \sin(\omega x)$, *then* $F(k) = \frac{\omega^k}{k!}\sin(\frac{\pi k}{2})$

Proof : By using the definition of differential transform

$$f(x) = \sum_{k=0}^{\infty} x^k F(k)$$

We have,

$$\omega x - \frac{(\omega x)^3}{3!} + \frac{(\omega\ x)^5}{5!} - \frac{(\omega x)^7}{7!} + ... = F(0) + xF(1) + x^2F(2) + x^3F(3) + ...$$

$$F(0) = F(2) = F(4) = F(6) = ...F(2k) = 0$$

$$F(1) = \omega, F(3) = -\frac{\omega^3}{3!}, F(5) = \frac{\omega^5}{5!}, ...F(2k+1) = (-1)^k \frac{\omega^(2k+1)}{(2k+1)!}$$

Thus we get $F(k) = \dfrac{\omega^k}{k!}\sin(\dfrac{\pi k}{2})$ $\square$

Theorem 14 *: If $f(x) = \cos(\omega x)$, then $F(k) = \frac{\omega^k}{k!}\cos\frac{\pi k}{2})$*

Proof : By using the definition of differential transform

$$f(x) = \sum_{k=0}^{\infty} x^k F(k)$$

$$\text{We have, } 1 - \frac{\omega^2}{2!}x^2 + \frac{\omega^4}{4!}x^4 - \frac{\omega^6}{6!}x^6 + ... = F(0) + xF(1) + x^2F(2) + x^3F(3) + ...$$

$$F(1) = F(3) = F(5) = F(7) = ...F(2k+1) = 0$$

$$\text{Then } F(0) = 1$$

$$F(2) = -\frac{\omega^2}{2!}$$

$$F(4) = \frac{\omega^4}{4!}$$

$$F(6) = -\frac{\omega^6}{6!}$$

$$\vdots$$

$$F(2k) = (-1)^k\frac{\omega^k}{k!}$$

$$\text{Finally, by substituting all we have } F(k) = \frac{\omega^k}{k!}\cos(\frac{\pi k}{2})$$ □

Chapter 2

Application to differential transform method for solving initial value problem of ordinary differential equations

Differential transform is widely applicable to solve many real life problems. Especially it is useful to solve initial-value problems, integral equations and system of differential equations. Here we can see some applications of differential transform solving ordinary differential equations.
In this section we will show how the differential transform can be used to solve initial-value problems of ordinary differential equations (i.e homogeneous, non homogeneous, variable coefficient, constant coefficient, linear and nonlinear). And the numerical solution is compared to the exact solution.

2.1 How to solve nonlinear function

Case 1. Exponential nonlinearity: $f(y) = e^{ay}$, $y = y(x)$
From the definition of the transform,

$$F(0) = \left[e^{ay(x)}\right]_{x=0} = e^{ay(0)} = e^{aY(0)} \tag{2.1}$$

Now, taking a differentiation of $f(y)$ with respect to x, we get:

$$\frac{df(y)}{dx} = ae^{ay}\frac{dy(x)}{dx} = af(y)\frac{dy(x)}{dx} \tag{2.2}$$

Application of the differential transform to Eq.(2.1) gives:

$$(k+1)F(k+1) = a\sum_{m=0}^{k}(m+1)Y(m+1)F(k-m) \tag{2.3}$$

Replacing k+1 by k gives:

$$F(k) = a\sum_{m=0}^{k-1}\frac{m+1}{k}Y(m+1)F(k-1-m), \quad k \geq 1 \tag{2.4}$$

Combining Eq.(2.1) and (2.4), we obtain the recursive relationship for calculating the function $f(y) = e^{ay}$:

$$F(k) = \begin{cases} e^{aY(0)} & \text{if } k = 0 \\ a\sum_{m=0}^{k-1}\frac{m+1}{k}Y(m+1)F(k-1-m) & \text{if } k \geq 1 \end{cases}$$

Case 2. Logarithmic nonlinearity: $f(y) = ln(a + by)$, $a + by > 0$.
By the definition of transform,

$$F(0) = \Big[ln(a + by(x))\Big]_{x=0} = ln(a + by(0)) = ln(a + bY(0)). \tag{2.5}$$

Further, differentiating $f(y) = ln(a + by)$ with respect to x, we get:

$$\frac{df(y(x))}{dx} = \frac{b}{a+by}\frac{dy(x)}{dx}, \tag{2.6}$$

Or equivalently,

$$a\frac{df(y)}{dx} = b\Big(\frac{dy(x)}{dx} - y\frac{df(y)}{dx}\Big) \tag{2.7}$$

Take the differential transform of Eq.(2.7) to get:

$$aF(k+1) = b\Big(Y(k+1) - \sum_{m=0}^{k}\frac{m+1}{k+1}F(m+1)Y(k-m)\Big) \tag{2.8}$$

Replacing k+1 by k yields:

$$aF(k) = b\Big(Y(k) - \sum_{m=0}^{k-1}\frac{m+1}{k}F(m+1)Y(k-1-m)\Big), \quad k \geq 1 \tag{2.9}$$

Substitute k=1 in to Eq.(2.9) to get:

$$F(1) = \frac{b}{a+bY(0)}Y(1). \tag{2.10}$$

For $k \geq 2, Eq.$(2.8) can be rewritten as:

$$F(k) = \frac{b}{a+bY(0)}\Big(Y(k) - \sum_{m=0}^{k-2}\frac{m+1}{k}F(m+1)Y(k-1-m)\Big), \; k \geq 2$$

$$F(k) = \begin{cases} ln(a+bY(0)) & \text{if } k = 0 \\ \frac{b}{a+bY(0)}Y(1) & \text{if } k = 1 \\ \frac{b}{a+bY(0)}\Big(Y(k) - \sum_{m=0}^{k-2}\frac{m+1}{k}F(m+1)Y(k-1-m)\Big) & \text{if } k \geq 2 \end{cases}$$

2.1.1 Procedure to solve IVPs using Differential transform:

Given the IVP

$$\begin{cases} F(x, y, y', y'', ...y^{(n)}) = 0. \\ y(x_0) = c_0, y'(x_0) = c_1, y''(x_0) = c_2, ..., y^{(n-1)}(x_0) = c_n. \end{cases}.$$

Then, to find $y(x)$ satisfying the IVP, we use the following procedures.
Step 1: The differential transform of each term in the given differential equation is computed
Step 2: The recurrence equation is obtained
Step 3: Y(0), Y(1), Y(2), Y(3), ... are calculated by the recurrence equation and given initial condition
Step 4: Finally, these values are substituted in to $y(x) = \sum_{k=0}^{\infty} x^k Y(k)$

Example 1: Let us consider the second order nonlinear initial-value problem:

$$y''(x) = 2y + ylny, \quad y > 0, \tag{2.11}$$

Subject to the initial condition

$$y(0) = 1, \;\; y'(0) = 0 \tag{2.12}$$

Solution: Taking the differential transform of Eq.(2.11), leads to

$$(k + 1)(k + 2)Y(k + 2) = 2Y(k) + 4\sum_{m=0}^{k} Y(m)F(k - m) \tag{2.13}$$

From the initial condition, given by Eq.(2.12)

$$Y(0) = 1, \quad Y(1) = 0 \tag{2.14}$$

Where $F(k)$ is the transform function of lny

$$F(k) = \begin{cases} ln(Y(0)) & \text{if } k = 0 \\ \frac{Y(1)}{Y(0)} & \text{if } k = 1 \\ \frac{Y(k)}{Y(0)} - \sum_{m=0}^{k-2} \frac{m+1}{kY(0)} F(m + 1)Y(k - 1 - m) & \text{if } k \geq 2 \end{cases} \tag{2.15}$$

Substituting Eq.(2.14) and $k = 0$ in to Eq.(2.15) and (2.13) to get

$$F(0) = 0, \quad Y(2) = 1 \tag{2.16}$$

Substitute Eq.(2.14) and (2.16) and $k = 1$ in to Eq.(2.15) and (2.13), we have:

$$F(1) = 0, \;\; Y(3) = 0$$

Following the same recursive procedure, we obtain:

$$Y(4) = \frac{1}{2!}, \; Y(5) = 0, \; Y(6) = \frac{1}{3!}, \; Y(7) = 0, \; Y(8) = \frac{1}{4!}$$

And so on. In general, we find:

$$Y(2k) = \frac{1}{k!}, \;\; Y(2k+1) = 0 \;\; k = 0, 1, 2, ...$$

Substituting all Y(k) in to the function $y(x) = \sum_{k=0}^{\infty} x^k Y(k)$, we obtained the exact solution as:

$$y(x) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + ... = \sum_{k=0}^{\infty} \frac{1}{k!}(x^2)^k = e^{x^2}$$

Example 2: Let us consider the third order non homogeneous ordinary differential equation.

$$y''' + 2y'' - y' - 2y = e^x, \qquad 0 \leq x \leq 3 \tag{2.17}$$

Subject to initial conditions

$$y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 0 \tag{2.18}$$

With exact solution

$$y(x) = \frac{43}{36}e^x + \frac{1}{4}e^{-x} - \frac{4}{9}e^{-2x} + \frac{1}{6}(xe^x)$$

Solution: Taking the differential transform of (2.17), leads to

$$y(k+3) = \frac{1}{(k+1)(k+2)(k+3)} \times \Big[k+1)y(k+1) - 2 \times (k+1)(k+2)y(k+2) + 2(k) + \frac{1}{k!}\Big] \tag{2.19}$$

From the initial condition given by Eq.(2.18), we obtain:

$$Y(0) = 1, \; Y(1)) = 2, \; Y(2) = 0 \tag{2.20}$$

Substitute Eq.(2.19) in to Eq.(2.20), and by recursive methods the results are listed as follow.

$$
\begin{aligned}
Y(3) &= \frac{5}{6} \\
Y(4) &= \frac{-5}{24} \\
Y(5) &= \frac{2}{15} \\
Y(6) &= \frac{-13}{360} \\
Y(7) &= \frac{59}{5040} \\
Y(8) &= \frac{-37}{13440} \\
Y(9) &= \frac{17}{30240} \\
Y(10) &= \frac{-1}{9072} \\
Y(11) &= \frac{87}{4435200} \\
&\vdots
\end{aligned}
$$

The solution is $y(x) = 1 + 2x + \frac{5}{6}x^3 - \frac{5}{24}x^4 + \frac{2}{15}x^5 - \frac{13}{360}x^6 + \frac{59}{5040}x^7 - \frac{37}{13440}x^8 + \frac{17}{30240}x^9 - \frac{1}{9072}x^{10} + \frac{87}{4435200}x^{11} - ...$

Octave Code:

$> x$=0:0.1:3;

$> f = 43./36 * exp(x) + 1./4 * exp(-x) - 4./9 * exp(-2 * x) + 1./6 * x. * exp(x);$

$> g = 1 + 2 * x + 5./6 * x.^3 - 5./24 * x.^4 + 2./15 * x.^5 - 13./360 * x.^6 + 59./5040 * x.^7$
$-37./13440 * x.^8 + 17./30240 * x.^9 + -1./9072 * x.^{10} + 87./4435200 * x.^{11};$

>plot(x,f,'r','linewidth',2,x,g,'o','linewidth',2)
> title('comparison between the exact solution and DTM')
>legend('exact', 'approximate(DTM)')
>ylabel('y(x)')
>xlabel('$0 \leq x \leq 3'$)

We obtain the following graph, that is the comparison of approximate and exact solution of the given differential equation depend on the order of expansion using Octave.
The line (graph) in the red color indicates the actual solution, while the ring line (o) indicates the approximate solution.

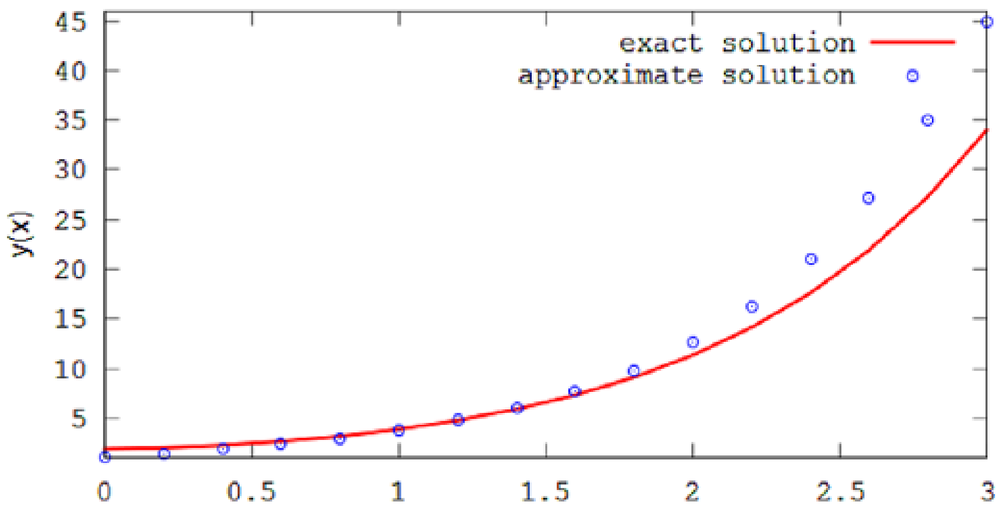

Figure 2.1: Comparison between exact solution and 5-order DTM

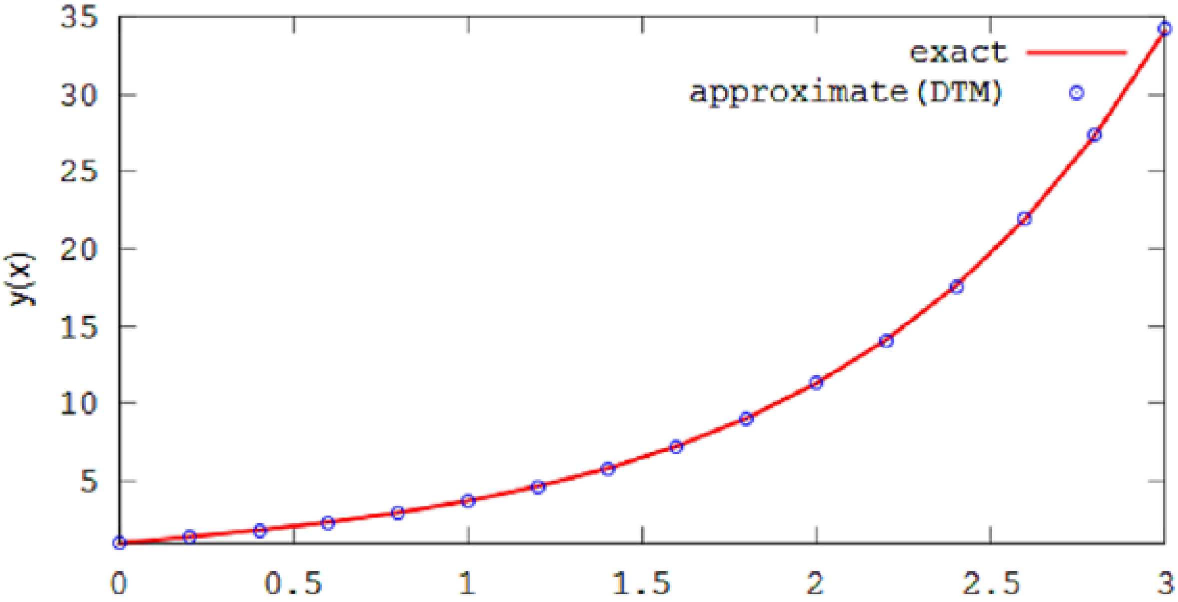

Figure 2.2: Comparison between exact solution and 11-order DTM

The above two graphs show that the comparison between the exact solution and 5 and 11 order of DTM. The first graph shows that the error is large for the 5^{th} order, but on getting 11^{th} order, the error is minimal for small values of x, by increasing the order of approximation more accuracy can be obtained.
Then, we can say that the DTM is more accurate and converges to the exact solution.

Example 3 Let us consider the second order non homogeneous ordinary differential equation.

$$y'' - 3y' + 2y = 2x - 3 \tag{2.21}$$

Subject to initial condition

$$y(0) =, \quad y'(0) = 2 \tag{2.22}$$

Solution: Taking the differential transform of Eq.(2.21), leads to

$$Y(k+2) = \frac{1}{(k+1)(k+2)} \times \Big[(k+1)Y(k+1) - 2Y(k) + 2\delta(k-1) - 3\delta(k)\Big] \tag{2.23}$$

From the initial condition given by Eq.(2.22) we have

$$Y(0) = 1, \;\; Y(1) = 2 \tag{2.24}$$

Substituting Eq.(2.23) into Eq.(2.24) and by recursive method, the results are listed as follows

$$\begin{aligned} Y(2) &= \frac{1}{2} = \frac{1}{2!} \\ Y(3) &= \frac{1}{6} = \frac{1}{3!} \\ Y(4) &= \frac{1}{24} = \frac{1}{4!} \\ Y(5) &= \frac{1}{120} = \frac{1}{5!} \end{aligned}$$

$$\vdots$$

Therefore the closed solution can be easily written as

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} x^k Y(k) \\ &= 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4}x^4 + ... \\ &= x + (1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4}x^4 + ...) \\ &= x + e^x \end{aligned}$$

Example 4 Let us consider the fourth order ordinary differential equation.

$$y^{(4)} = e^x \qquad 0 \leq x \leq 1. \tag{2.25}$$

With initial conditions

$$y(0) = 3, \; y'(0) = 1 \;, y''(0) = 5 \; and \; y'''(0) = 1 \tag{2.26}$$

The exact solution by integration and evaluation integral constant is

$$\begin{aligned} y(x) &= 2 + 2x^2 + e^x = 2 + 2x^2 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \\ &= 3 + \frac{5}{2}x^2 + \frac{1}{3!}x^3 + ... \end{aligned}$$

Solution: From the initial condition Eq.(2.26) we get

$$Y(0) = 3,\ Y(1) = 1,\ Y(2) = \frac{5}{2}\ and\ Y(3) = \frac{1}{6} \tag{2.27}$$

Applying the differential transform Eq.(2.25) we get,

$$Y(k+4) = \frac{1}{k!(k+1)(k+2)(k+3)(k+4)} \tag{2.28}$$

Substituting Eq.(2.28) into Eq.(2.27) and by recursive method, the results are listed as follows

$$\begin{aligned}
Y(4) &= \frac{1}{24}\\
Y(5) &= \frac{1}{120}\\
Y(6) &= \frac{1}{720}\\
Y(7) &= \frac{1}{5040}\\
Y(8) &= \frac{1}{40320}
\end{aligned}$$

Substituting all $Y(k)$ in to the function $y(x) = \sum_{k=0}^{\infty} x^k Y(k)$, we obtained the series solution as the following.

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) = Y(0)+Y(1)x+x^2Y(2)+x^3Y(3)+x^4Y(4)+x^5Y(5)+x^6Y(6)+x^7Y(7)+x^8Y(8)+...$$

$$y(x) = 3 + x + \frac{5}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8$$

x	Exact	DTM	DTM error
0.1	3.1252	3.1252	0.0000
0.2	3.3014	3.3014	0.0000
0.3	3.5299	3.5299	0.0000
0.4	3.8118	3.8118	0.0000
0.5	4.1487	4.1487	0.0000
0.6	4.5421	4.5421	0.0000
0.7	4.9938	4.9938	0.0000
0.8	5.5055	5.5055	0.0000
0.9	6.0796	6.0796	0.0000
1.0	6.7183	6.7183	0.0000

Table 2.1: Numerical result of example 4

Table 2.1 shows the comparison between results of DTM and the exact solution computed for 8^{th} order and selected values of x on the given interval. And also table 2.1 gives the absolute errors between the the exact results and DTM.
We can see from table 2.1, that the errors are minimal (zero) for small values of x.
We can then say that DTM is more accurate and converges to the exact solution.

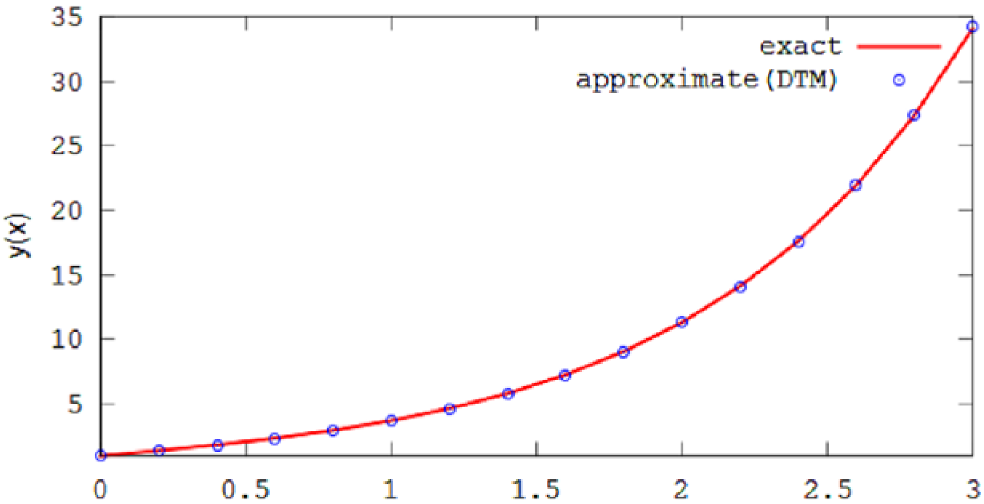

Figure 2.3: Comparison between exact solution and DTM

Example 5: Let us consider second order differential equation with variable coefficient

$$\begin{cases} y'' + y - z'' - 4z = 0. \\ y' + z' = \cos(x) + 2\cos(2x). \end{cases} \tag{2.29}$$

With the conditions

$$\begin{cases} y(0) = 0, y'(0) = 1. \\ z(0) = 0, z'(0) = 2. \end{cases} \tag{2.30}$$

The exact solution of this problem is $y(x) = \sin(x),\ z(x) = \sin(2x)$
Solution: Applying Differential Transform, we have

$$\begin{cases} (k+1)(k+2)Y(k) + Y(k) - (k+1)(k+2)Z(k+2) - 4Z(k) = 0. \\ (k+1)Y(k+1) + (k+1)Z(k+1) = \frac{1}{k!}\cos(\frac{k\pi}{2}) + \frac{2^{k+1}}{k!}\cos(\frac{k\pi}{2}). \\ Y(0) = 0, Y(1) = 1 \\ Z(0) = 0, Z(1) = 2 \end{cases}$$

Consequently, we find

$$Y(2) = 0, Z(2) = 0$$
$$Y(3) = \frac{-1}{3!}, Z(3) = \frac{-8}{3!}$$
$$Y(4) = 0, Z(4) = 0$$
$$Y(5) = \frac{1}{5!}, Z(5) = \frac{32}{5!}$$

Therefore, the solution is given by

$$y(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 = sin(x)$$
$$z(x) = 2x - \frac{8}{3!} + \frac{32}{5!}x^5 = sin(2x)$$

The rate of change $\frac{dp}{dt} = birth - death + immigration - migration$ is a conservation equation for the population.
Where
p_0 = initial population (population you that with) at time $t = 0$,
r= relative growth rate that is constant
t = the time the population grows.
$p(t)$ = the population of the species at time t

2.2 Solution of the Malthus model for population growth

The simplest model (the Malthus model for population growth) has no immigration and migration and thus the birth and death terms are proportional to p (linear). That is $\frac{dp}{dt} = bp - dp$, subject to the initial condition $p(0) = p_0$ where Mathematically b and d are positive constant and the biological meaning of b and d respectively are birth rate and death rate of the species p.

Let us consider the first order simple growth model

$$\frac{dp}{dt} = rp \text{ where } r = b - d \tag{2.31}$$

Subject to the initial condition

$$p(0) = p_0 \tag{2.32}$$

Where $\frac{dp}{dt}$ is the rate of change of the population and $p(t)$ be the number of individual in a population at a time t

Solution: Taking the differential transform of Eq.(2.31), leads to

$$(k+1)P(k+1) = rP(k) \Rightarrow P(k+1) = \frac{1}{k+1}\Big[rP(k)\Big] \tag{2.33}$$

From the initial condition, given by Eq.(2.32)

$$P(0) = p_0, \tag{2.34}$$

is obtained. Substituting Eq.(2.33) in to Eq.(2.34) and by recursive method, the results are listed as follows

$$\begin{aligned}
P(1) &= rp_0, \\
P(2) &= \frac{1}{2}r^2 p_0 \\
P(3) &= \frac{1}{3!}r^3 p_0 \\
P(4) &= \frac{1}{4!}r^4 p_0 \\
P(5) &= \frac{1}{5!}r^5 p_0 \\
P(6) &= \frac{1}{6}r^6 p_0 \\
&\vdots
\end{aligned}$$

Substituting all P(k) in to the function $p(t) = \sum_{k=0}^{\infty} t^k P(k)$, we obtained the series solution as the following

$$\begin{aligned}
p(t) &= \sum_{k=0}^{\infty} t^k P(k) \\
&= p_0 + p_0 rt + \frac{1}{2!}p_0(rx)^2 + \frac{1}{3!}(rx)^3 p_0 + \frac{1}{4!}(xr)^4 p_0 + \frac{1}{5!}(xr)^5 p_0 + \frac{1}{6!}(xr)^6 p_0 ..., \\
&= p_0(1 + rt + \frac{1}{2!}(rx)^2 + \frac{1}{3!}(rx)^3 + \frac{1}{4!}(xr)^4 + \frac{1}{5!}(xr)^5 + \frac{1}{6!}(xr)^6 \\
&= p_0 e^{rt}
\end{aligned}$$

- Here we are not interested negative values of p since it now represents the quantity that has to be positive to have biological relevance i.e population size.

- If the death rate exceeds the birth rate, then $r < 0$
  Mathematically:
  As $t \to \infty$, $p(t) \to 0$, meaning for long t, the number of human population decreases.

- If the death rate equal to the birth rate, then $r = 0$
  Mathematically:
  As $t \to \infty$, $p(t)$=constant, meaning for long time t the population is unchanged

- However, if $r > 0$ then the population exponential growth Mathematically:
  As $t \to \infty$, $p(t) \to \infty$, meaning for long time t, the population increases indefinitely which is unrealistic

That result is unrealistic because as p becomes sufficiently large other factors will be will undoubtedly come in to play, such as insufficient food or other resources.

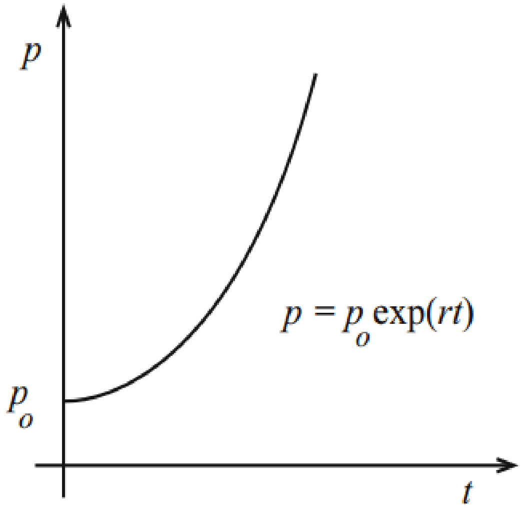

Figure 2.4: The Malthus model for population growth: $p(t) = p_0 e^{rt}$

- The population of a community is known to increase at a rate proportional to the number of people present at a time t. If the population has doubled in 6 years, how long it will take to triple?

Solution: Let p(t) denote the population at time t. Let $p(0)$ denote the initial population (population at $t = 0$).

$$\begin{aligned}
p(t) &= Ae^{rt}, \text{ where } A = p(0)\\
Ae^{6r} &= p(6) = 2p(0) = 2A\\
e^{6r} &= 2 \text{ or } r = \frac{1}{2}ln2\\
\text{Find t when } p(t) &= 3A = 3p(0)\\
p(0)e^{rt} &= 3p(0) \text{ or}\\
3 &= e^{\frac{1}{6}ln(2)t} \text{ or}\\
ln(3) &= \frac{(ln2)t}{6} \text{ or}\\
t &= \frac{6ln3}{ln2} \approx \text{ 9.6 years approximately 9 years 6 months}
\end{aligned}$$

Example 6: Let us consider nonlinear initial value problem

$$y'' + 2(y')^2 + 8y = 0 \tag{2.35}$$

With initial values

$$y(0)=0, \quad y'(0)=1 \tag{2.36}$$

Using the definition of differential transform, we get the following recursive relation:

$$Y(k+2)=-\frac{1}{(k+1)(k+2)}\Big[2\sum_{m=0}^{k}(m+1)(k-m+1)Y(m+1)Y(k-m+1)+8Y(k)\Big]. \tag{2.37}$$

For $k=0$ we have: $Y(2)=-\frac{1}{2}\Big[2(Y(1))^2+8Y(0)\Big]=-1,$
For $k=1: \ Y(3)=-\frac{1}{6}[2(2Y(1)Y(2)+2Y(2)Y(1))+8Y(1)\Big]=0$
For $k=2: \ Y(4)=-\frac{1}{6}[2(3Y(1)Y(3)+4(Y(2))^2+3Y(3)Y(1))+8Y(2)\Big]=0$
Using the same procedure finally we have, $Y(k)=0$ for $k=3,4,...$, the exact solution of the initial value problem is:

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^k=Y(0)+Y(1)x+Y(2)x^2=x-x^2$$

Example 7:
Consider the initial value problem

$$y''=\frac{8y^2}{1+2x} \tag{2.38}$$

$$y(0)=1, \ y'(0)=-2 \tag{2.39}$$

Eq.(2.38) can be written as

$$y''+2xy''+8f(y)=0, \ f(y)=y^2 \tag{2.40}$$

The differential transform is then applied to obtain

$$(k+1)(k+2)Y(k+2)+2\sum_{m=0}^{k}(k-m+1)(k-m+2)\delta(m-1)Y(k-m+2)-8F(k)=0 \tag{2.41}$$

Where F(k), are given by

$$\begin{aligned}
&F(0)=Y^2(0),\\
&F(1)=2Y(1)Y(0)\\
&F(2)=Y^2(1)+2Y(0)Y(2),\\
&F(3)=2[Y(0)Y(3)+Y(1)Y(2)],\\
&F(4)=Y^2(2)+2[Y(0)Y(4)+Y(1)Y(3)].
\end{aligned}$$

In this initial value problem we have the transformed initial conditions accordingly:

$$Y(0)=1, Y(1)=2, \tag{2.42}$$

In view of the recurrence scheme (2.41) and (2.42) the following results are obtained for k = 0, 1, 2, . . . , 8:

$$\begin{aligned}
&Y(0) = 1,\ Y(5) = -32\\
&Y(1) = -2,\ Y(6) = 64\\
&Y(2) = 4,\ Y(7) = -128\\
&Y(3) = -8,\ Y(8) = 256\\
&Y(4) = 16,
\end{aligned}$$

The series solution is now given by

$$y = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - 32x^5 + 64x^6 - 128x^7 + 256x^8 + ...,$$

Which gives the exact solution: $y = \sum_{k=0}^{\infty}(-2x)^k = \frac{1}{1+2x}$

Conclusion

In this paper, the Differential Transformation Method (DTM) has been successfully applied to find exact and approximate solution of the first, second, third and fourth order initial value problems of ordinary differential equations (variable coefficient, constant coefficient, homogeneous and non homogeneous). First, some fundamental theorems of DTM are provided and then used to solve initial value problems of ODEs. We have obtained approximate analytical solution of the given problem.
If the approximate solution of the given problems are compared with their analytical solutions, the differential transform method is very effective and convergence are quite close. It may be concluded that DTM is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear differential equations.

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